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The power radiated by two correlated sources

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Abstract

We analyze the total power that is radiated by two correlated point sources. The influence of the degree of coherence between the two sources and of the distance between them can clearly be distinguished. Significant modulations of the total radiated power are predicted. Both primary and secondary sources are investigated.

1. Introduction

When the light from two point sources interferes, as, for example, in Thomas Young's celebrated experiment [1,2], the visibility of the interference fringes that are formed is a direct measure of the correlation between the source fields [3]. The *spectral interference law* expresses how the spectral density (or 'spectral intensity') at an observation point in the region of superposition depends on the *spectral degree of coherence* [4, Sec. 4.3.2].

In this paper we study the total power that is radiated by two correlated point sources. As we will demonstrate, the power is modulated significantly by both the distance between the two sources, and by their spectral degree of coherence. Especially when the distance between the two sources is of the order of a wavelength, a situation that is commonly found in acoustics, strong modulations are predicted. Further testimony to the relevance of our study is that after completion of the manuscript we discovered its resemblance to a more restricted research problem suggested in Ref. [4] of which the solution has never been published. Our analysis treats both primary and secondary sources. We illustrate our findings with numerical results.

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2. Primary sources

Consider two identical, small primary sources located at points Q_1 and Q_2, that are separated by a vector $\mathbf{d}$. Let $S_Q(\omega)$ be the spectrum of each source, ω being an angular frequency. The field at an observation point P is then given by the formula

$$U(P,\omega) = U(Q_1,\omega)\frac{e^{ikR_1}}{R_1} + U(Q_2,\omega)\frac{e^{ikR_2}}{R_2}, \tag{1}$$

where R_i is the distance between Q_i and P ($i = 1,2$), and $k = \omega/c$ with c the speed of light in vacuum. If the spectral degree of coherence between the two source fields is denoted by $\mu_Q(\omega)$, then the spectral density at P is given by

$$S(P,\omega) = \langle U^*(P,\omega)U(P,\omega)\rangle, \tag{2}$$

$$= S_Q(\omega)\left\{\frac{1}{R_1^2} + \frac{1}{R_2^2} + \left[\mu_Q(\omega)\frac{e^{ik(R_2-R_1)}}{R_1R_2} + \text{c.c.}\right]\right\}. \tag{3}$$

Here c.c. is the complex conjugate, and the angle brackets denote an average taken over an ensemble of source field realizations (in the sense of coherence theory in the space-frequency domain [4, Ch. 4]) that are assumed to be stationary, at least in the wide sense. The spectral degree of coherence of the two source fields satisfies the relation

$$\mu_Q(\omega) = \frac{W_Q(Q_1,Q_2;\omega)}{S_Q(\omega)}, \tag{4}$$

with

$$W_Q(Q_1, Q_2; \omega) = \langle U^*(Q_1, \omega) U(Q_2, \omega) \rangle \tag{5}$$

being the cross-spectral density of the two source fields. When the point P is in the far-zone, we have to a good approximation (see Fig. 1),

$$\frac{e^{ik(R_2 - R_1)}}{R_1 R_2} \approx \frac{e^{ik\mathbf{d}\cdot\mathbf{s}_2}}{R_2^2}, \tag{6}$$

with $\mathbf{s}_2$ a unit vector. On substituting from Eq. (6) into Eq. (3) we obtain for the far-zone spectrum the formula

$$S^{(\infty)}(P, \omega) = \frac{S_Q(\omega)}{R_2^2} \left\{ 2 + \left[\mu_Q(\omega) e^{ik\mathbf{d}\cdot\mathbf{s}_2} + \text{c.c.} \right] \right\}. \tag{7}$$

Suppressing the subscript 2 and writing $S^{(\infty)}(R\mathbf{s}, \omega)$ rather than $S^{(\infty)}(P, \omega)$, Eq. (7) becomes

$$S^{(\infty)}(R\mathbf{s}, \omega) = \frac{S_Q(\omega)}{R^2} \left\{ 2 + \left[\mu_Q(\omega) e^{ik\mathbf{d}\cdot\mathbf{s}} + \text{c.c.} \right] \right\}. \tag{8}$$

On integrating this result over all possible angles we find for the total power that is radiated by two correlated primary point sources the expression

$$P(\omega) = R^2 \int_{(4\pi)} S^{(\infty)}(R\mathbf{s}, \omega)\, d\Omega, \tag{9}$$

$$= 2S_Q(\omega) \left\{ 4\pi + \left[\frac{1}{2} \mu_Q(\omega) \int_{(4\pi)} e^{ik\mathbf{d}\cdot\mathbf{s}}\, d\Omega + \text{c.c.} \right] \right\}, \tag{10}$$

$$= 8\pi S_Q(\omega) \left[1 + j_0(kd) \Re \mu_Q(\omega) \right], \tag{11}$$

where $\Re$ denotes the real part, $d = |\mathbf{d}|$ and j_0 is the spherical Bessel function of the first kind and of order zero. It is seen from Eq. (11) that the total power that is radiated at frequency ω consists of the sum of the contributions of the two sources, and an interference term which depends on both the spectral degree of coherence of the two source fields, $\mu_Q(\omega)$, and on the separation between the two sources.

Let us now discuss the implications of Eq. (11) for two limiting cases. First, when the separation between the two sources is much smaller than the wavelength, i.e., when $d \ll \lambda$, then

$$j_0(kd) \approx 1, \tag{12}$$

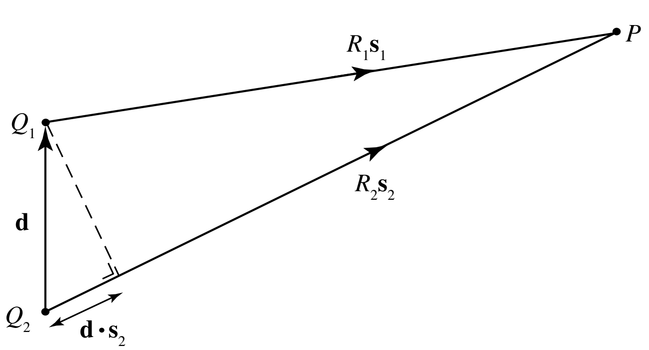

Fig. 1. Illustrating the notation relating to the far-zone approximation (6).

and Eq. (11) gives

$$P(\omega) \approx 8\pi S_Q(\omega)[1 + \Re \mu_Q(\omega)]. \tag{13}$$

Since

$$-1 \leqslant \Re \mu_Q(\omega) \leqslant 1, \tag{14}$$

we see that in this case

$$0 \leqslant P(\omega) \leqslant 16 S_Q(\omega). \tag{15}$$

If the two sources are completely uncorrelated, i.e., if $\mu_Q(\omega) = 0$, then, according to Eq. (11),

$$P(\omega)_{\text{uncorr.}} = 8\pi S_Q(\omega). \tag{16}$$

Hence we may rewrite the inequality (15) as

$$0 \leqslant P(\omega) \leqslant 2P(\omega)_{\text{uncorr.}}. \tag{17}$$

In the case when $P(\omega) = 0$ there is obviously complete cancelation of the far-field due to destructive interference. Since $S^{(\infty)}(R\mathbf{s}, \omega) \geqslant 0$, this implies, according to Eq. (9), that $S^{(\infty)}(R\mathbf{s}, \omega) = 0$ for all directions s. So the system behaves as a *non-radiating source* [5]. In the other limiting case, when $P(\omega) = 2P(\omega)_{\text{uncorr.}}$, there is an increase in the radiated power due to constructive interference and, consequently the far-field spectral density in the far-zone averaged over all directions is increased.

In the limit when the separation between the two sources is much larger than the wavelength, i.e., when $d \gg \lambda$, then

$$j_0(kd) \approx 0, \tag{18}$$

and Eq. (11) gives

$$P(\omega) \approx P(\omega)_{\text{uncorr.}}. \tag{19}$$

It is seen from Eq. (19) that for separation distances much greater than the wavelength, the total radiated power is independent of the spectral degree of coherence of the fields generated by the two sources.

Fig. 2 shows the behavior of the total power radiated by the two sources as a function of their separation distance d (in units of wavelengths) and the real part of their spectral

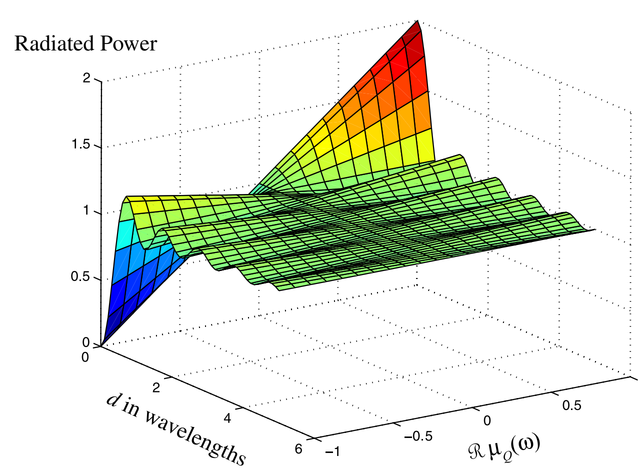

Fig. 2. The normalized power $P(\omega)/P(\omega)_{\text{uncorr.}}$ radiated by two primary sources as a function of their separation distance d, and the real part of their spectral degree of coherence $\Re \mu_Q(\omega)$.

degree of coherence $\Re\mu_Q(\omega)$. Especially when the source separation is of the order of a wavelength (a situation that is commonly found, for example, in acoustics) both parameters strongly modulate the radiated power.

3. Secondary sources

Consider an opaque screen Q, located in the plane $z = 0$, with two pinholes of radius a centered at the points specified by the vectors $\mathbf{x} = (d/2, 0, 0)$ and $-\mathbf{x}$ (see Fig. 3). The secondary source field in each aperture is $U_i(\omega)$, with $i = 1,2$. Again the spectral densities are assumed to be equal, namely $S_Q(\omega)$. The field at the screen is thus given by the expression

$$U_Q(\boldsymbol{\rho}, \omega) = U_1(\omega)\mathrm{circ}(\boldsymbol{\rho} + \mathbf{x}) + U_2(\omega)\mathrm{circ}(\boldsymbol{\rho} - \mathbf{x}), \tag{20}$$

where $\boldsymbol{\rho} = (x, y, 0)$ denotes a position vector of a point in the plane $z = 0$, and

$$\mathrm{circ}(\boldsymbol{\rho}) = \begin{cases} 1 & |\boldsymbol{\rho}| \leqslant a \\ 0, & |\boldsymbol{\rho}| > a. \end{cases} \tag{21}$$

The cross-spectral density of the field in the plane of the screen is, therefore, given by the expression

$$W_Q(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \langle [U_1^*(\omega)\mathrm{circ}(\boldsymbol{\rho}_1 + \mathbf{x}) + U_2^*(\omega)\mathrm{circ}(\boldsymbol{\rho}_1 - \mathbf{x})] \times [U_1(\omega)\mathrm{circ}(\boldsymbol{\rho}_2 + \mathbf{x}) + U_2(\omega)\mathrm{circ}(\boldsymbol{\rho}_2 - \mathbf{x})] \rangle, \tag{22}$$

The *radiant intensity* of the far-field in a direction indicated by the unit vector $\mathbf{s}$ is related to this cross-spectral density by the equation [4, Eq. (5.3–8)]

$$J(\mathbf{s}, \omega) = \left(\frac{k}{2\pi}\right)^2 \cos^2\theta \times \iint_{-\infty}^{\infty} W_Q(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) \times \exp[-\mathrm{i}k\mathbf{s}_\perp \cdot (\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)] \mathrm{d}^2\rho_1 \, \mathrm{d}^2\rho_2, \tag{23}$$

where θ is the angle that $\mathbf{s}$ makes with the positive z-axis, and $\mathbf{s}_\perp = (s_x, s_y)$. On substituting from Eq. (22) into Eq. (23) we find that

$$J(\mathbf{s}, \omega) = a^2 J_1^2(ka)\cos^2\theta S_Q(\omega) \times \{2 + [\mu_Q(\omega)\exp(-\mathrm{i}ks_x d) + \mathrm{c.c.}]\}. \tag{24}$$

It is seen from Eq. (24) that for source radii a such that $J_1(ka) = 0$ the radiant intensity vanishes identically. It is to be noted that this not due to interference between the two sources, because in this case both sources themselves are non-radiating. The observation that for certain sizes a coherent source does not radiate at all was first made by Carter and Wolf [6].

The total power, at frequency ω, that is radiated by the sources is given by the formula [4, Eq. 5.7–53]

$$P(\omega) = \int_0^{2\pi}\int_0^{\pi/2} J(\mathbf{s}, \omega)\sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi. \tag{25}$$

Now $s_x = \sin\theta\cos\phi$; on using this expression and also that

$$3\int_0^{\pi/2} J_0(kd\sin\theta)\cos^2\theta\sin\theta \,\mathrm{d}\theta = 3\left[\frac{\sin kd - kd\cos kd}{(kd)^3}\right], \tag{26}$$

$$= j_0(kd) + j_2(kd), \tag{27}$$

we find for the total power that is radiated at frequency ω by two correlated secondary sources the expression

$$P(\omega) = \frac{4\pi}{3}a^2 J_1^2(ka) \times S_Q(\omega)\{1 + [j_0(kd) + j_2(kd)]\Re\mu_Q(\omega)\}, \tag{28}$$

Regarding Eq. (28) it is of interest to remark that $-1 \leqslant j_0(x) + j_2(x) \leqslant 1$.

The structure of Eq. (11), which pertains to primary sources, and Eq. (28), which pertains to secondary sources, is rather similar. One difference is that the dependence on the source separation distance is through a single spherical Bessel function of the first kind in one case, and through a sum of two such functions in the other case. The behavior of these functions is illustrated in Fig. 4. It is seen that the interference term for two primary sources has zeros for different values of the separation distance than the interference term for two secondary sources. However, the behavior of the total power that is radiated in the limiting cases of $d \ll \lambda$ and $d \gg \lambda$ is the same for pairs of sources of both types.

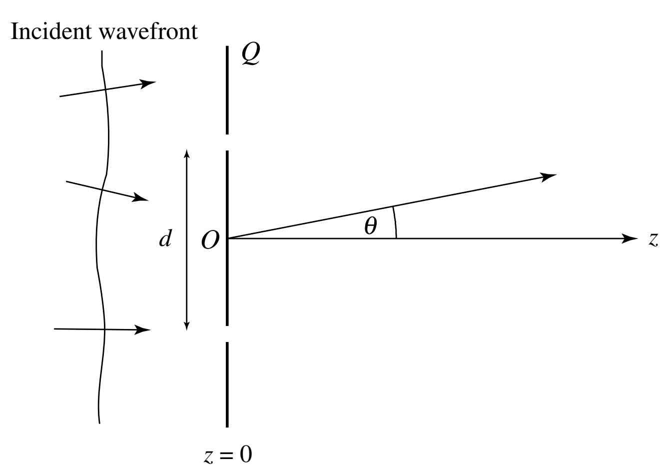

Fig. 3. Two secondary sources.

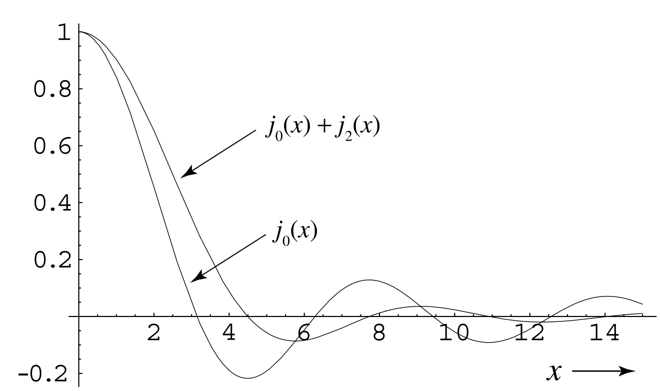

Fig. 4. Spherical Bessel functions of the first kind.

In conclusion, we have derived expressions for the total power that is radiated by a pair of correlated, primary and secondary sources. The expressions consist of a direct contribution from each source and an interference term. The latter is the product of two factors: the real part of the spectral degree of coherence, and an oscillating function of the distance between the sources. When the separation distance is small compared to the wavelength – a situation that is commonly met in acoustics – the modulating influence of the degree of coherence and of the separation distance can be quite significant.

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