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An energy theorem for scattering of partially coherent beams

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Abstract

A recent generalization of the optical cross-section theorem is applied to obtain an expression for the rate at which energy is removed by scattering and absorption from a wide class of partially coherent beams. The class of beams considered includes the outputs of many lasers. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

A basic result of scattering theory is the so-called optical cross-section theorem (usually called the optical theorem) [1]. It expresses the rate at which energy is removed from a plane wave incident on a medium by the processes of scattering and absorption. Since the development of the laser in the 1960's, many scattering experiments have been and are being performed with laser beams rather than with plane waves. In this note we apply a recently derived generalization of the optical cross-section theorem to determine the rate at which energy is removed from an incident field which is a beam rather than a plane wave. Our results apply to a broad class of beams, the so-called Gaussian–Schell-model beams. This class includes the outputs of many lasers. We illustrate the results by examples.

2. The generalized optical cross-section theorem

We begin by briefly reviewing the traditional formulation of the optical cross-section theorem and of a recent generalization of it.

The usual formulation applies to situations where a field, $\Psi^{(i)}(\boldsymbol{r},t)$, incident on a deterministic scattering object is a monochromatic plane wave of unit amplitude, propagating in a direction specified by a real unit vector $\boldsymbol{s}_0$,

$$\Psi^{(i)}(\boldsymbol{r},t) = \psi^{(i)}(\boldsymbol{r})\,\mathrm{e}^{-i\omega t}, \tag{2.1}$$

with

$$\psi^{(i)}(\boldsymbol{r}) = \mathrm{e}^{ik\boldsymbol{r}\cdot\boldsymbol{s}_0}, \tag{2.2}$$

where

$$k = \omega/c, \tag{2.3}$$

c being the speed of light in vacuum. Assuming that one deals with static scattering, i.e. that the macroscopic physical properties of the scattering medium do not change with time, the scattered field will also be monochromatic and of the same

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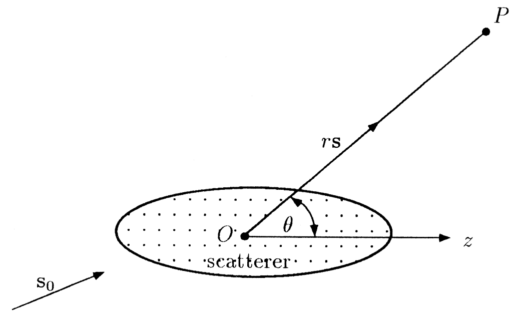

Fig. 1. Illustrating notation relating to scattering. $\boldsymbol{s}_0$ and $\boldsymbol{s}$ are real unit vectors in the directions of incidence and scattering respectively.

frequency ω and its space-dependent part at a point P at a distance r in the far zone, in the direction specified by a unit vector $\boldsymbol{s}$ (see Fig. 1), has the asymptotic form

$$\psi^{(s)}(r\boldsymbol{s}) \sim f(\boldsymbol{s},\boldsymbol{s}_0)\frac{\mathrm{e}^{ikr}}{r}. \tag{2.4}$$

($kr \to \infty$, $\boldsymbol{s}$ fixed). The extinction cross-section, $\sigma^{(e)}$ may then be expressed in the form [1]

$$\sigma^{(e)} = \frac{4\pi}{k}\mathrm{Im}\{f(\boldsymbol{s}_0,\boldsymbol{s}_0)\}, \tag{2.5}$$

where ‘Im’ denotes the imaginary part. This theorem asserts that the power extinguished by scattering and by absorption, expressed in units of the power incident on the scatterer per unit area perpendicular to the direction of incidence, $\boldsymbol{s}_0$, is proportional to the scattering amplitude $f(\boldsymbol{s}_0,\boldsymbol{s}_0)$ in the forward direction.

In a recent paper [2] we presented a generalization of this theorem to situations where the incident field is a statistically stationary random field of any state of spatial coherence. More specifically the incident field was taken to be represented by a statistical ensemble of which each member was a superposition of plane wave modes propagating into the half-space $z > 0$, i.e.

$$\psi^{(i)}(\boldsymbol{r}) = \int_{\boldsymbol{s}_\perp^2 \le 1} a(\boldsymbol{s}_\perp)\,\mathrm{e}^{ik\boldsymbol{r}\cdot\boldsymbol{s}}\,\mathrm{d}^2\boldsymbol{s}_\perp. \tag{2.6}$$

In this formula $\boldsymbol{s}$ is again a real unit vector with Cartesian components

$$\boldsymbol{s} \equiv (s_x,s_y,s_z), \quad (s_z \ge 0), \tag{2.7}$$

and

$$\boldsymbol{s}_\perp \equiv (s_x,s_y,0) \tag{2.8}$$

is the projection, considered as a two-dimensional vector, of the unit vector $\boldsymbol{s}$ onto the plane $z = 0$. The generally complex amplitudes $a(\boldsymbol{s}_\perp)$ are random variables. It was shown in Ref. [2] that with such an incident field the total extinguished power, i.e. the rate at which energy is removed from the incident field by scattering and absorption, is given by the formula

$$P^{(e)} = \frac{4\pi}{k}\int\int \mathrm{Im}\{\mathscr{A}(\boldsymbol{s}_\perp,\boldsymbol{s}'_\perp)\, f(\boldsymbol{s},\boldsymbol{s}')\}\,\mathrm{d}^2s'_\perp\,\mathrm{d}^2s_\perp, \tag{2.9}$$

where

$$\mathscr{A}(\boldsymbol{s}_\perp,\boldsymbol{s}'_\perp) = \langle a^*(\boldsymbol{s}_\perp)a(\boldsymbol{s}'_\perp)\rangle \tag{2.10}$$

is the angular correlation function of the incident random field [3]. In Eq. (2.10) the asterisk denotes complex conjugate and the angular brackets denote the average taken over the ensemble of the $a(\boldsymbol{s}_\perp)$. The integrations on the right hand side of Eq. (2.9) are taken over the domains $\boldsymbol{s}_\perp^2 \le 1$ and $\boldsymbol{s}'^2_\perp \le 1$.

3. The optical cross-section theorem for scattering of Gaussian–Schell-model beams

An important class of fields of different states of spatial coherence is the class of so-called Gaussian–Schell-model beam. We will now apply the generalized cross-section theorem (2.9) to investigate scattering of beams of this kind.

For a Gaussian–Schell-model beam the spectral density $S^{(0)}(\boldsymbol{\rho})$ and the spectral degree of coherence $\mu^{(0)}(\boldsymbol{r}_1,\boldsymbol{r}_2)$, [4], p. 171, at fixed frequency ω in the plane of the waist of the beam ($z=0$) are given by the expressions

$$S^{(0)}(\boldsymbol{\rho}) = A_0^2 \exp\left(-\rho^2/2\sigma_s^2\right), \tag{3.1}$$

and

$$\mu^{(0)}(\boldsymbol{r}_1,\boldsymbol{r}_2)|_{z=0} \equiv g^{(0)}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) = \exp\left[-(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^2/2\sigma_g^2\right] \tag{3.2}$$

respectively. In these formulae $\boldsymbol{\rho}$, $\boldsymbol{\rho}_1$, and $\boldsymbol{\rho}_2$ are two-dimensional vectors which represent points in the plane $z=0$ and A_0, σ_s, and σ_g are positive constants. Evidently $\mu^{(0)} \to 1$ as $\sigma_g \to \infty$ and the expressions (3.1) and (3.2) then represent a distribution in the waist of a lowest-order Hermite–Gaussian laser mode.

The angular correlation function of the field generated by the distributions (3.1) and (3.2) can be shown to be given by the expression

$$\mathscr{A}(\boldsymbol{s}_{1\perp},\boldsymbol{s}_{2\perp}) = \left(\frac{k^2 A_0^2}{4\pi}\right) \Delta^2 \sigma_s^2 \exp\left\{\tfrac{1}{8}\left[-4k^2\sigma_s^2(\boldsymbol{s}_{1\perp}+\boldsymbol{s}_{2\perp})^2 - k^2\Delta^2(\boldsymbol{s}_{1\perp}-\boldsymbol{s}_{2\perp})^2\right]\right\}, \tag{3.3}$$

where

$$\frac{1}{\Delta^2} = \frac{1}{4\sigma_s^2} + \frac{1}{\sigma_g^2}. \tag{3.4}$$

The angular distribution of the radiant intensity produced by the beam is given by the expression (cf. [4], Eq. 5.6-53)

$$J(\theta) = \Delta^2 \sigma_s^2 A_0^2 \pi \exp\left(-2k\sigma_s^2 \sin^2\theta\right), \tag{3.5}$$

where θ is the angle which the unit vector $\boldsymbol{s}$ pointing to the point P in the far zone makes with the positive z-axis (see Fig. 1). Evidently, in order that the field generated by the boundary conditions (3.1) and (3.2) is a beam it is necessary that $k\sigma_s \gg 1$.

An expression for the total power extinguished on scattering of a Gaussian–Schell-model beam is obtained at once on substituting the expression (3.3) for the angular correlation function into the generalized optical theorem (2.9). One then obtains the formula

$$P^{(e)} = \frac{k^3}{4\pi} A_0^2 \Delta^2 \sigma_s^2 \int\int \mathrm{Im}\left\{\exp\left[\frac{-4k^2\sigma_s^2(\boldsymbol{s}_{1\perp}+\boldsymbol{s}_{2\perp})^2 - k^2\Delta^2(\boldsymbol{s}_{1\perp}-\boldsymbol{s}_{2\perp})^2}{8}\right] f(\boldsymbol{s}_1,\boldsymbol{s}_2)\right\} \mathrm{d}^2 s_{1\perp}\, \mathrm{d}^2 s_{2\perp}. \tag{3.6}$$

The scattering amplitude $f(\boldsymbol{s}_1,\boldsymbol{s}_2)$ in the integrand on the right-hand side of Eq. (3.6) depends, of course, on the nature of the scatterer. To illustrate the use of this formula let us assume that the scatterer is spherically symmetric and that the axis of the incident beam passes through the center of the scatterer. The scattering amplitude then has the form

$$f(\boldsymbol{s}_1,\boldsymbol{s}_2) \equiv h(\boldsymbol{s}_1 \cdot \boldsymbol{s}_2), \tag{3.7}$$

i.e., the scattering amplitude depends only on the angle between the direction of scattering $\boldsymbol{s}_1$ and direction of incidence $\boldsymbol{s}_2$.

The contribution to the integral in Eq. (3.6) is evidently only appreciable when the values of s_{1x}, s_{1y}, s_{2x}, and s_{2y}, are small compared to both $(k\Delta)^{-1}$ and to $(k\sigma_s)^{-1}$. Recalling the definition (3.4) of Δ this implies that as σ_s and σ_g increase the contribution to the total extinguished power comes mainly from forward scattering. Increasing the value of σ_s implies that the effective cross-section of the beam in the waist plane $z=0$ becomes larger; increasing σ_g implies that the spatial coherence area of the field in that plane also increases. As both $\sigma_s \to \infty$ and $\sigma_g \to \infty$ the field approaches a coherent plane wave and the usual form of optical theorem for plane waves then follows from Eq. (3.6).

These remarks may be put in a more quantitative form. For this purpose we expand the scattering amplitude in a Taylor series in two variables for small values of $|s_{1\perp}|$ and $|s_{2\perp}|$:

$$\begin{aligned} h(\boldsymbol{s}_1 \cdot \boldsymbol{s}_2) = {} & h(1) + s_{1x} h_{1x}(1) + s_{1y} h_{1y}(1) + s_{2x} h_{2x}(1) + s_{2y} h_{2y}(1) + \frac{s_{1x}^2}{2} h_{1x,1x}(1) + \frac{s_{1y}^2}{2} h_{1y,1y}(1) \\ & + \frac{s_{2x}^2}{2} h_{2x,2x}(1) + \frac{s_{2y}^2}{2} h_{2y,2y}(1) + s_{1x}s_{1y} h_{1x,1y}(1) + s_{1x}s_{2x} h_{1x,2x}(1) + s_{1x}s_{2y} h_{1x,2y}(1) \\ & + s_{1y}s_{2x} h_{1y,2x}(1) + s_{1y}s_{2y} h_{1y,2y}(1) + s_{2x}s_{2y} h_{2x,2y}(1) + \ldots, \end{aligned} \tag{3.8}$$

where the subscripts denote partial differentiation with respect to the components of the unit vectors, i.e.

$$h_{1x}(1) = \left.\frac{\partial h(\boldsymbol{s}_1 \cdot \boldsymbol{s}_2)}{\partial s_{1x}}\right|_{\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = 1}, \quad h_{1x,2y}(1) = \left.\frac{\partial^2 h(\boldsymbol{s}_1 \cdot \boldsymbol{s}_2)}{\partial s_{1x} \partial s_{2y}}\right|_{\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = 1} \ldots, \tag{3.9}$$

etc.

Because of azimuthal symmetry about the direction of the beam axis implied by the assumed spherical symmetry of the scatterer, all the first derivatives of the scattering amplitude with respect to the components of the unit vectors must be identically zero when evaluated for the forward direction. If this were not so, the scattered field would have a cusp in that direction which would contradict known analytic properties of the scattering amplitude [5]. If $h'(x) \equiv \mathrm{d}h(x)/\mathrm{d}x$ the terms in expansion (3.8) which involve second partial derivatives of the scattering amplitude may be written in a more compact form as

$$h_{1x,1x}(1) = h_{1y,1y}(1) = h_{2x,2x}(1) = h_{2y,2y}(1) = -h'(1), \tag{3.10}$$

$$h_{1x,2x}(1) = h_{1y,2y}(1) = h'(1) \tag{3.11}$$

and all the other terms involving second partial derivatives are identically zero. The expansion (3.8) can then be written as

$$h \approx h(1) - \frac{(\boldsymbol{s}_{1\perp} - \boldsymbol{s}_{2\perp})^2}{2} h'(1). \tag{3.12}$$

On substituting from Eq. (3.12) into Eq. (3.6) (recalling Eq. (3.7)), one finds that the extinguished power is given by the expression

$$\begin{aligned} P^{(e)} \approx & \frac{k^3}{4\pi} A_0^2 \Delta^2 \sigma_s^2 \int\int \exp\left[-2\sigma_s^2 k^2 (\boldsymbol{s}_{1\perp} + \boldsymbol{s}_{2\perp})^2\right] \\ & \times \exp\left[\frac{-(k\Delta)^2 (\boldsymbol{s}_{1\perp} - \boldsymbol{s}_{2\perp})^2}{8}\right] \mathrm{Im}\left\{h(1) - \frac{(\boldsymbol{s}_{1\perp} - \boldsymbol{s}_{2\perp})^2}{2} h'(1)\right\} \mathrm{d}^2 s_{1\perp}\, d^2 s_{2\perp}. \end{aligned} \tag{3.13}$$

Carrying out the integration one finds that

$$P^{(e)} = A_0^2 \frac{4\pi}{k} \mathrm{Im}\left\{h(1) - \frac{1}{(k\Delta)^2} h'(1) + \mathrm{O}\left[(k\Delta)^{-3}\right]\right\}, \tag{3.14}$$

or, in terms of our original notation for the scattering amplitude,

$$P^{(e)} = A_0^2 \frac{4\pi}{k} \mathrm{Im}\left\{f(\boldsymbol{s}_0, \boldsymbol{s}_0) + \frac{1}{(k\Delta)^2} \left.\frac{f(\boldsymbol{s}_1, \boldsymbol{s}_2)}{\partial s_{1x} \partial s_{1x}}\right|_{\boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}_0} + \mathrm{O}\left[(k\Delta)^{-3}\right]\right\}. \tag{3.15}$$

On comparing Eq. (3.15) with the usual form of the optical theorem expressed by Eq. (2.5) we see that the first term on the right hand side of Eq. (3.15) is precisely what one would obtain if a monochromatic plane wave of amplitude A_0 was incident on the scatterer in the direction specified by $\boldsymbol{s}_0$. (In that case the extinction cross-section $\sigma^{(e)}$ and the extinguished power $P^{(e)}$ are, of course, related by the formula $A_0^2 \sigma^{(e)} = P^{(e)}$.) The second term on the right hand side of Eq. (3.15) evidently represents a correction term, in the leading order of the parameter $(k\Delta)^{-1}$, arising from the beam nature of the incident field. We see that the correction term depends on the second derivative with respect to the scattering angle of the scattering amplitude in the forward direction.

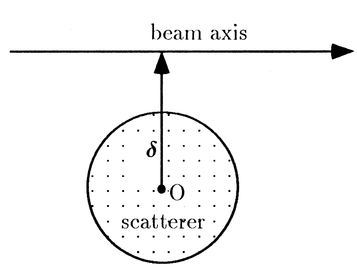

Fig. 2. Illustrating the significance of the displacement vector $\boldsymbol{\delta}$ for scattering of a Gaussian–Schell-model beam by a spherically symmetric medium when the beam axis does not pass through the center of the scatterer.

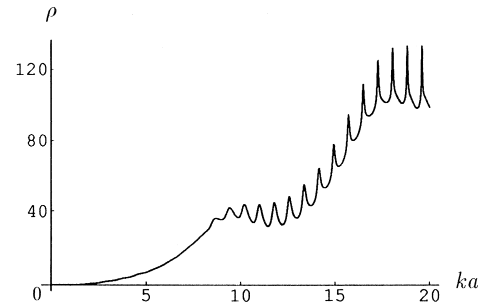

Fig. 3. The factor ρ, defined by Eq. (3.19) versus the size parameter ka for scattering on a homogeneous sphere when the index of refraction of the sphere is $n=1.4$, with $k\Delta \gg 1$.

This example applies to the situation when the beam axis passes through the center of the spherically symmetric scatterer. Suppose now that the beam axis is displaced with respect to the center of the scatterer. Let δ denote the vectorial distance between the axis of the beam and the center of the scatterer, O (see Fig. 2). The angular correlation function, $\mathscr{A}_\delta(\boldsymbol{s}_1,\boldsymbol{s}_2)$, of this displaced beam, refered to the same coordinate axes as before (with the origin O at the center of the scatterer), can readily be shown, for example, by application of the shift theorem for Fourier transforms, to be related to the angular correlation function (3.3) by the formula

$$\mathscr{A}_\delta(\boldsymbol{s}_{1\perp},\boldsymbol{s}_{2\perp}) = \mathscr{A}(\boldsymbol{s}_{1\perp},\boldsymbol{s}_{2\perp})\exp[-ik(\boldsymbol{s}_{1\perp}-\boldsymbol{s}_{2\perp})\cdot\boldsymbol{\delta}]. \tag{3.16}$$

Retaining the assumption of spherical symmetry of the scatterer and making use of the approximation (3.12) one may evaluate the two leading terms in the expansion of the extinguished power given by Eq. (3.6) and one finds that

$$P^{(e)} \approx A_0^2 \frac{4\pi}{k}\exp\left(\frac{-2\delta^2}{\Delta^2}\right)\mathrm{Im}\left\{h(1) - \frac{1}{(k\Delta)^2}h'(1) + \mathrm{O}\left[(k\Delta)^{-3}\right]\right\}. \tag{3.17}$$

Comparison of Eqs. (3.17) and (3.14) shows that the only change in the extinguished power arising from the displacement of the beam axis away from the center of the scatterer is the presence of the factor $\exp(-2\delta^2/\Delta^2)$. Consequently the extinguished power decreases exponentially with the square of the scaled offset distance δ/Δ.

Returning to formula (3.14) pertaining to the case where the beam axis passes through the center of the scatterer we see that if terms of order $(k\Delta)^{-3}$ are neglected, the ratio of the power extinguished from a Gaussian–Schell-model beam of peak amplitude A_0 to the power extinguished from a plane wave of amplitude A_0 is given by the expression

$$P^{(e)}_{\text{beam}}/P^{(e)}_{\text{plane wave}} = 1 - \frac{\rho}{(k\Delta)^2}, \tag{3.18}$$

where

$$\rho \equiv \frac{\mathrm{Im}\, h'(1)}{\mathrm{Im}\, h(1)}. \tag{3.19}$$

In the Appendix this quantity is evaluated for some typical cases. Fig. 3 represents the factor ρ as a function of the size parameter, ka, of the scatterer, for a homogeneous sphere of refractive index $n=1.4$. We note the rapid increase of ρ when

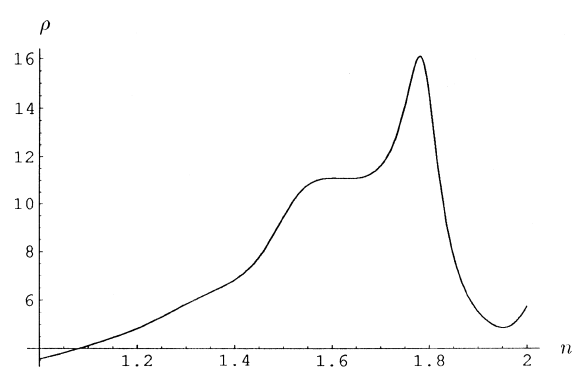

Fig. 4. The factor ρ, defined by Eq. (3.19), versus the index of refraction n for scattering on a homogeneous sphere when the size parameter of the sphere $ka=5$, and $k\Delta \gg 1$.

the size parameter exceeds unity and the appearance of resonance peaks for larger values of ρ. The dependence of the ρ on the index of refraction is shown in Fig. 4.

The plots in Figs. 3 and 4 were made from data generated by the computer program Mathematica from the formulae in the Appendix. The series converged very rapidly when the number of terms retained exceeded the size parameter ka. The curves were calculated with 50 terms in the series.

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Appendix A. Calculations of the factor ρ, defined by Eq. (3.19), for a homogeneous sphere

The scattering amplitude for scattering of a plane wave on a homogeneous sphere of radius a is known to be given by the expression (Ref. [6], p. 932)

$$h(x) = \frac{1}{k}\sum_{m=0}^{\infty}(2m+1)\alpha_m P_m(x), \tag{A1}$$

where $P_m(x)$ are the Legendre polynomials, and the coefficients α_m, which depend on the index of refraction of the sphere, are given by the equation

$$\alpha_m = \frac{i\beta_m}{\beta_m - i\gamma_m}, \tag{A2}$$

with parameters β_m and γ_m being given by the formulae

$$\beta_m = kj_m(k'a)j'_m(ka) - k'j'_m(k'a)j_m(ka), \tag{A3}$$

$$\gamma_m = k'j'_m(k'a)n_m(ka) - j_m(k'a)n'_m(ka). \tag{A4}$$

In Eqs. (A3) and (A4), j_m and n_m denote the spherical Bessel functions and spherical Neumann functions respectively and k' is the wavenumber inside the sphere, i.e.

$$k' = nk. \tag{A5}$$

The scattering amplitude and the first derivative of the scattering amplitude in the forward direction $(x=1)$ with respect to the argument are then given by the formulae

$$h(x)|_{x=1} = \frac{1}{k}\sum_{m=0}^{\infty}(2m+1)\alpha_m,$$

$$h'(x)|_{x=1} = \frac{1}{k}\sum_{m=0}^{\infty}(2m+1)\frac{m(m+1)}{2}\alpha_m.$$

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