Generalized optical theorem for reflection, transmission, and extinction of power for scalar fields

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We present a derivation of the optical theorem that makes it possible to obtain expressions for the extinguished power in a very general class of problems not previously treated. The results are applied to the analysis of the extinction of power by a scatterer in the presence of a lossless half space. Applications to microscopy and tomography are discussed.

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I. INTRODUCTION

The conservation of energy for electromagnetic or acoustic waves, or conservation of probability in quantum mechanics, leads to a remarkable identity known as the optical theorem. The optical theorem relates the power extinguished from a plane wave incident on an object to the scattering amplitude in the direction of the incident field. Explicitly, the optical theorem may be expressed by the formula [1–5]

$$\sigma_e = \frac{4\pi}{k_0} \operatorname{Im} A(\mathbf{k},\mathbf{k}), \tag{1}$$

where σ_e is the extinction cross section, k_0 is the wave number of the field, $\mathbf{k}$ is the wave vector of the incident plane wave, and $A(\mathbf{k},\mathbf{k})$ is the amplitude of the field scattered in the forward direction, i.e., in the direction of the incident field. The extinction cross section is the total extinguished power (i.e., the power lost to scattering and absorption of the incident field) normalized by the power per unit area incident on the scatterer. This theorem implies that the total power extinguished from the incident field is removed from the incident field by means of interference between the incident field and the forward scattered field.

The optical theorem, expressed by Eq. (1), may be seen to be a special case of a more general theorem that applies to the scattering of an arbitrary incident field and relates the extinguished power to a weighted integral of the scattering amplitude [6,7]. This result accounts for the contribution not only of the homogeneous components but also of the evanescent components of the incident field which have to be taken into account when the source of illumination is in the near zone of the scatterer.

The generalized optical theorem also applies to the scattering of partially coherent fields and scattering from random objects. It has been shown that the generalized optical theorem may be used to relate extinguished power to the structure of the scattering object and even to reconstruct the spatial structure of the scatterer from the data obtained from power extinction experiments [8,9].

In Sec. II, we derive an expression for extinguished power that is the progenitor of the generalized optical theorem. An expression is obtained that relates the extinguished power to a volume integral over the domain of the scatterer. Unlike other forms of the optical theorem, this result is obtained without invoking the asymptotic behavior of the scattered field. Furthermore, this formulation allows for the case that the scattering object is embedded in an arbitrary background with inhomogeneous optical properties. In Sec. III, scattering from an object in free space is reconsidered and the equivalence of our result and the generalized optical theorem [6,7] is established. In Sec. IV, the problem of scattering from an object in a half space is addressed. It is found that the extinguished power is related to the field that is scattered in the directions of the components of the incident field in both half spaces. The half-space problem is of practical importance in imaging and tomography when the sample is supported on a slide or other flat platform.

II. GENERAL RESULTS

Consider the reduced wave equation for a monochromatic, scalar field $\psi(\mathbf{r})e^{-i\omega t}$, $\omega=ck_0$, in the presence of a scattering object described by a susceptibility $\eta(r)$, embedded in a non-absorbing background medium, characterized by the spatially dependent wave number $k(\mathbf{r})$,

$$\nabla^2\psi(\mathbf{r}) + k^2(\mathbf{r})\psi(\mathbf{r}) = -4\pi k^2(\mathbf{r})\eta(\mathbf{r})\psi(\mathbf{r}). \tag{2}$$

Equation (2) is ubiquitous in physics. It is encountered, for example, as the linear, single particle, time independent, nonrelativistic Schrödinger equation [[10], Sec. 15]. It is also the governing equation for the propagation of sound waves [[11], Sec. 6.2].

Assuming that the background is nonabsorbing [$k(\mathbf{r})$ is real], the reduced wave equation (2) then implies that

$$\psi^*(\mathbf{r})\nabla^2\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla^2\psi^*(\mathbf{r}) = -4\pi k(\mathbf{r})^2\{\eta(\mathbf{r}) - \eta^*(\mathbf{r})\}|\psi(\mathbf{r})|^2. \quad (3)$$

By integrating both sides of Eq. (3) over any volume V which is bounded by a closed surface S with unit outward normal $\mathbf{n}$, and applying Green's theorem [[12], Sec. 1.8], it is readily found that

$$\int_S d^2r\,\{\psi^*(\mathbf{r})\,\nabla\,\psi(\mathbf{r}) - \psi(\mathbf{r})\,\nabla\,\psi^*(\mathbf{r})\}\cdot\mathbf{n}dS = -\int_V d^3r 4\pi k(\mathbf{r})^2\{\eta(\mathbf{r}) - \eta^*(\mathbf{r})\}|\psi(\mathbf{r})|^2. \quad (4)$$

In the classical wave theory, the expression in the curly brackets { } on the left hand side of Eq. (4) is usually identified with the energy flux density of the field [[11], p. 199]; in quantum mechanics such an expression is identified with the probability current density [[10], Sec. 17]. The flux density vector [[5], Appendix 11], with normalization taken so that a unit amplitude plane wave has a unit magnitude flux density vector, is thus defined as

$$\mathbf{F}(\mathbf{r}) = \frac{1}{2ik_0}\{\psi^*(\mathbf{r})\,\nabla\,\psi(\mathbf{r}) - \psi(\mathbf{r})\,\nabla\,\psi^*(\mathbf{r})\}. \quad (5)$$

The absorbed power is given by the net flux passing through any surface enclosing the scatterer (but not enclosing the sources of the incident field):

$$P_a = -\int_S d^2r\,\mathbf{F}(\mathbf{r})\cdot\mathbf{n}, \quad (6)$$

where P_a is the power absorbed by the scattering medium. It can be seen from Eq. (4) that P_a is also given by the expression

$$P_a = \frac{4\pi}{k_0}\,\mathrm{Im}\int_V d^3r\,k^2(\mathbf{r})|\psi(\mathbf{r})|^2\eta(\mathbf{r}), \quad (7)$$

where Im denotes the imaginary part. When η is real, the scatterer is nonabsorbing and P_a is identically zero.

A distinction between the background medium and the scattering object is made on physical grounds, although the effect of the scatterer itself might also be included in the function $k(\mathbf{r})$. The distinction is physically important because the field is taken to be composed of two parts, a scattered field ψ_s and an incident field ψ_i with $\psi_i + \psi_s = \psi$. The incident field may be identified as the field which is present in the absence of the scattering object. The incident field evidently obeys the Helmoltz equation

$$\nabla^2\psi_i(\mathbf{r}) + k^2(\mathbf{r})\psi_i(\mathbf{r}) = 0, \quad (8)$$

except in the region occupied by the source, and it is assumed that the source is located outside the domain occupied by the scatterer. From Eqs. (2) and (8) it is seen that the scattered field ψ_s obeys the equation

$$\nabla^2\psi_s(\mathbf{r}) + k^2(\mathbf{r})\psi_s(\mathbf{r}) = -4\pi k(\mathbf{r})^2\eta(\mathbf{r})\psi(\mathbf{r}). \quad (9)$$

By analogy with Eq. (5), the flux density vector of the scattered field $\mathbf{F}_s$ is defined as

$$\mathbf{F}_s(\mathbf{r}) = \frac{1}{2ik_0}\{\psi_s^*(\mathbf{r})\,\nabla\,\psi_s(\mathbf{r}) - \psi_s(\mathbf{r})\,\nabla\,\psi_s^*(\mathbf{r})\}. \quad (10)$$

The power carried by the scattered field is then given by the expression

$$P_s = \int_S d^2r\,\mathbf{F}_s(\mathbf{r})\cdot\mathbf{n}, \quad (11)$$

where S is any closed surface which completely encloses the scattering volume. It may be seen that

$$\psi_s^*(\mathbf{r})\nabla^2\psi_s(\mathbf{r}) - \psi_s(\mathbf{r})\nabla^2\psi_s^*(\mathbf{r}) = -8\pi i k^2(\mathbf{r})\mathrm{Im}\{\eta(\mathbf{r})\psi_s^*(\mathbf{r})\psi(\mathbf{r})\}. \quad (12)$$

Making use of Eqs. (9) and (12), the power carried by the scattered field, Eq. (11), may also be expressed in terms of the volume integral as

$$P_s = -\frac{4\pi}{k_0}\,\mathrm{Im}\int_V d^3r\,k^2(\mathbf{r})\psi_s^*(\mathbf{r})\psi(\mathbf{r})\eta(\mathbf{r}). \quad (13)$$

The absorbed power and the scattered power are supplied by the incident field. The sum of the absorbed and the scattered power, referred to as the extinguished power

$$P_e = P_a + P_s, \quad (14)$$

represents the power depleted from the incident field because of the presence of the scattering body. From Eqs. (7), (13), and (14) it can be seen that the extinguished power is given in terms of the incident and the scattered fields by the expression

$$P_e = \frac{4\pi}{k_0}\,\mathrm{Im}\int_V d^3r\,k^2(\mathbf{r})\psi_i^*(\mathbf{r})\psi(\mathbf{r})\eta(\mathbf{r}). \quad (15)$$

Formula (15) is our main result. It gives the power depleted from the incident beam as a consequence of interference between the incident and the scattered fields in the domain of the object.

III. FREE SPACE

We will now show that, when the scatterer is located in free space, the general result expressed in Eq. (15) reduces to results derived previously in Refs. [6,7]. Moreover, the discussion will provide a template for the calculations of the subsequent section where the more difficult problem of scattering in a half space will be considered.

Let us choose a coordinate system such that the scattering object is located in the half space $z \geqslant 0$ and the sources of the incident field are located in the half space $z < 0$. The most general case, where sources may be located anywhere outside of the object, may be treated by an obvious extension the analysis presented here.

The incident field may be expressed in the form of an angular spectrum of plane waves [[13], Sec. 3.2],

$$\psi_i(\mathbf{r}) = \int d^2k_\parallel\, a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{16}$$

where $\mathbf{k}_\parallel$ is a vector parallel to the plane $z=0$ and $\mathbf{k}=\mathbf{k}_\parallel+\hat{\mathbf{z}}k_z$, with $\hat{\mathbf{z}}$ being the unit vector in the increasing z direction and

$$k_z = \sqrt{k_0^2 - k_\parallel^2}. \tag{17}$$

When the modulus $|\mathbf{k}_\parallel|$ exceeds the free-space wave number, k_z becomes purely imaginary. Such values of k_z correspond to modes of the incident field that decay exponentially on propagation and are known as evanescent waves. Since the system under consideration is linear in the fields, the total field is given by the expression

$$\psi(\mathbf{r}) = \int d^2k_\parallel\, a(\mathbf{k}) \phi(\mathbf{r};\mathbf{k}), \tag{18}$$

where $\phi(\mathbf{r},\mathbf{k})$ is the total field produced at a point $\mathbf{r}$ due to scattering of a plane wave of unit amplitude $[a(\mathbf{k})=1]$ with wave vector $\mathbf{k}$. The total field ϕ may be separated into incident and scattered fields,

$$\phi(\mathbf{r};\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}} + \phi_s(\mathbf{r};\mathbf{k}), \tag{19}$$

$\phi_s(\mathbf{r};\mathbf{k})$ being the scattered field produced at a point $\mathbf{r}$ on scattering of a plane wave of unit amplitude with wave vector $\mathbf{k}$, satisfying the integral equation

$$\phi_s(\mathbf{r};\mathbf{k}) = k_0^2 \int d^3r'\, G(\mathbf{r},\mathbf{r}')\, \eta(\mathbf{r}')\, \phi(\mathbf{r}';\mathbf{k}). \tag{20}$$

In Eq. (20) G is the retarded (outgoing) Green's function, which satisfies the equation

$$\nabla^2 G(\mathbf{r},\mathbf{r}') + k_0^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{21}$$

$\delta^{(3)}$ denoting the three-dimensional Dirac delta function. The Green's function may be represented as a superposition of plane wave modes,

$$G(\mathbf{r},\mathbf{r}') = \frac{i}{2\pi} \int \frac{d^2k_\parallel}{k_z} \exp[i\mathbf{k}_\parallel\cdot(\mathbf{r}-\mathbf{r}') + ik_z|z-z'|]. \tag{22}$$

The scatterer lies entirely in the region $z<z_0$ as shown in Fig. 1. The field ϕ_s, in the region $z>z_0$, may be represented in the form of an angular spectrum of plane waves

$$\phi_s(\mathbf{r};\mathbf{k}_1) = \frac{i}{2\pi} \int \frac{d^2k_{2\parallel}}{k_{2z}} A(\mathbf{k}_1,\mathbf{k}_2) e^{i\mathbf{k}_2\cdot\mathbf{r}}, \tag{23}$$

where A is the usual scattering amplitude between states of real momenta, as it may be seen that in the far zone of the scatterer, for $z>0$, the scattered field ϕ_s takes the asymptotic form

$$\phi_s(\mathbf{r};\mathbf{k}_1) \sim \frac{e^{ik_0r}}{r} A(\mathbf{k}_1, k_0\hat{\mathbf{r}}),\ \hat{\mathbf{r}} = \mathbf{r}/r. \tag{24}$$

It should be noted that, while A gives the asymptotic behavior of the field, it is not defined only by the asymptotic behavior of the field. It is also well defined for complex wave vectors which are relevant in the near field for evanescent plane wave modes. An explicit form for A may be found by considering the field in any plane $z=z_0$ (see Fig. 1) such that the scatterer lies entirely in the region $z<z_0$,

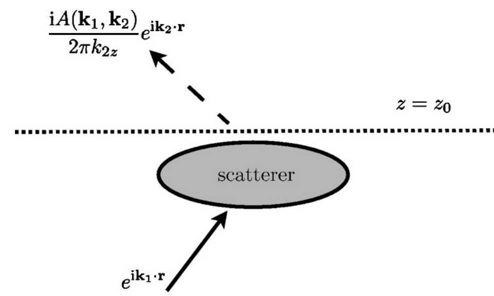

FIG. 1. Illustrating the notation for scattering in free space. The incident field, here taken to be a single plane wave, is represented by a solid line in the direction of the wave vector of the plane wave. A plane wave component of the scattered field is represented by a dashed line and is labeled with the appropriate amplitude.

$$A(\mathbf{k}_1,\mathbf{k}_2) = \frac{k_{2z}}{2\pi i} \int_{z=z_0} d^2r\, e^{-i\mathbf{k}_2\cdot\mathbf{r}} \phi_s(\mathbf{r};\mathbf{k}_1). \tag{25}$$

Substitution of the expression for ϕ_s given in Eq. (20) in the right hand side of Eq. (25) and use of Eq. (22) yields the relation

$$A(\mathbf{k}_1,\mathbf{k}_2) = k_0^2 \int_V d^3r\, e^{-i\mathbf{k}_2\cdot\mathbf{r}} \eta(\mathbf{r}) \phi(\mathbf{r};\mathbf{k}_1). \tag{26}$$

We now return to the extinguished power. Substituting Eqs. (16) and (18) in Eq. (15) we find that

$$P_e = 4\pi k\, \mathrm{Im} \int d^2k_{1\parallel} d^2k_{2\parallel}\, a(\mathbf{k}_1) a^*(\mathbf{k}_2) \times \int_V d^3r\, e^{-i\mathbf{k}_2^*\cdot\mathbf{r}} \eta(\mathbf{r}) \phi(\mathbf{k}_1,\mathbf{r}). \tag{27}$$

Thus, making use of Eq. (26), the results originally presented in Refs. [6,7] are recovered:

$$P_e = \frac{4\pi}{k_0} \mathrm{Im} \int d^2k_{1\parallel} d^2k_{2\parallel}\, a(\mathbf{k}_1) a^*(\mathbf{k}_2) A(\mathbf{k}_1,\mathbf{k}_2^*). \tag{28}$$

The case of illumination by a single plane wave of amplitude a may be recovered by taking $a(\mathbf{k})=a\delta^{(2)}(\mathbf{k}_\parallel-\mathbf{k}_{0\parallel})$. One finds that

$$P_e = \frac{4\pi|a|^2}{k_0} \mathrm{Im}\, A(\mathbf{k}_0,\mathbf{k}_0^*). \tag{29}$$

This expression was originally derived for real $\mathbf{k}$ by Feenberg [1] in the context of quantum mechanics, with the restrictions that $P_a=0$ and with implied cylindrical symmetry. An heuristic argument was given by van de Hulst [2] for the theorem in the context of electromagnetic scattering. A rigorous derivation for the electromagnetic case was given by Jones [3]. When the incident field is a plane wave of amplitude a, it is common practice to normalize the power by the intensity of the incident field. This normalized quantity, defined by the expression $\sigma_e = P_e/|a|^2$, and Eq. (1), is referred to as the extinction cross section [[4] Sec. 1.3]. For real $\mathbf{k}$, Eq. (1) is the *optical (cross section) theorem.*

IV. HALF SPACE

We now consider the problem of scattering from an object in the presence of a planar interface separating two homogeneous half spaces. It is assumed that the scatterer is of finite extent and is located entirely in the region $z_1 \geq z \geq z_2$ as shown in Fig. 2. The half space $z > 0$ is assumed to be vacuum. The other half space $z < 0$ is taken to consist of a material whose index of refraction is $n > 1$. The fields satisfy Eqs. (2) and (9), with

$$k(\mathbf{r}) = \begin{cases} k_0 & \text{for } z \geq 0, \\ nk_0 & \text{for } z < 0. \end{cases} \tag{30}$$

The Green's function $G(\mathbf{r},\mathbf{r}')$ for this situation satisfies the equation

$$\nabla^2 G(\mathbf{r},\mathbf{r}') + n^2(z)k_0^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{31}$$

and obeys the boundary conditions

$$G(\mathbf{r},\mathbf{r}')\big|_{z=0^+} = G(\mathbf{r},\mathbf{r}')\big|_{z=0^-}, \tag{32}$$

$$\hat{\mathbf{z}} \cdot \nabla G(\mathbf{r},\mathbf{r}')\big|_{z=0^+} = \hat{\mathbf{z}} \cdot \nabla G(\mathbf{r},\mathbf{r}')\big|_{z=0^-}. \tag{33}$$

$G(\mathbf{r},\mathbf{r}')$ admits the plane wave decomposition

$$\begin{aligned} G(\mathbf{r},\mathbf{r}') = \frac{\mathrm{i}}{2\pi}\int \frac{\mathrm{d}^2k_\parallel}{k_z}\exp[\mathrm{i}\mathbf{k}_\parallel\cdot(\mathbf{r}-\mathbf{r}')]\Big(\Theta(z)\Theta(z') \\ \times\{\exp(\mathrm{i}k_z|z-z'|) + R(\mathbf{k},\mathbf{k}')\exp[\mathrm{i}k_z(z'+z)]\} \\ + \Theta(-z)\Theta(z')T(\mathbf{k},\mathbf{k}')\exp[\mathrm{i}k_z z' - \mathrm{i}k_z' z] \\ + \Theta(z)\Theta(-z')\frac{k_z}{k_z'}T'(\mathbf{k},\mathbf{k}')\exp[\mathrm{i}k_z z - \mathrm{i}k_z' z'] \\ + \Theta(-z)\Theta(-z')\frac{k_z}{k_z'}\{\exp(\mathrm{i}k_z'|z-z'|) \\ + R'(\mathbf{k},\mathbf{k}')\exp[-\mathrm{i}k_z'(z'+z)]\}\Big), \end{aligned} \tag{34}$$

where Θ is the Heaviside step function, i.e., $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x \leq 0$. The factors R and T are the reflection and transmission coefficients, respectively,

$$R(\mathbf{k},\mathbf{k}') = \frac{k_z - k_z'}{k_z + k_z'},\ T(\mathbf{k},\mathbf{k}') = \frac{2k_z}{k_z + k_z'}, \tag{35}$$

$$R'(\mathbf{k},\mathbf{k}') = -R(\mathbf{k},\mathbf{k}'),\ \text{and}\ T'(\mathbf{k},\mathbf{k}') = \frac{k_z'}{k_z}T(\mathbf{k},\mathbf{k}'), \tag{36}$$

with the wave vectors, in the upper and lower half spaces, respectively, given by Eq. (17), and

$$\mathbf{k}' = \mathbf{k}_\parallel + \hat{\mathbf{z}}k_z',\ \text{and}\ k_z' = \sqrt{n^2k_0^2 - k_\parallel^2}. \tag{37}$$

The validity of Eq. (34) may be verified by direct substitution. It might be noted that the chosen notation for the reflection and transmission coefficients is somewhat redundant and not as compact as it might be. However, this expanded notation will prove useful shortly.

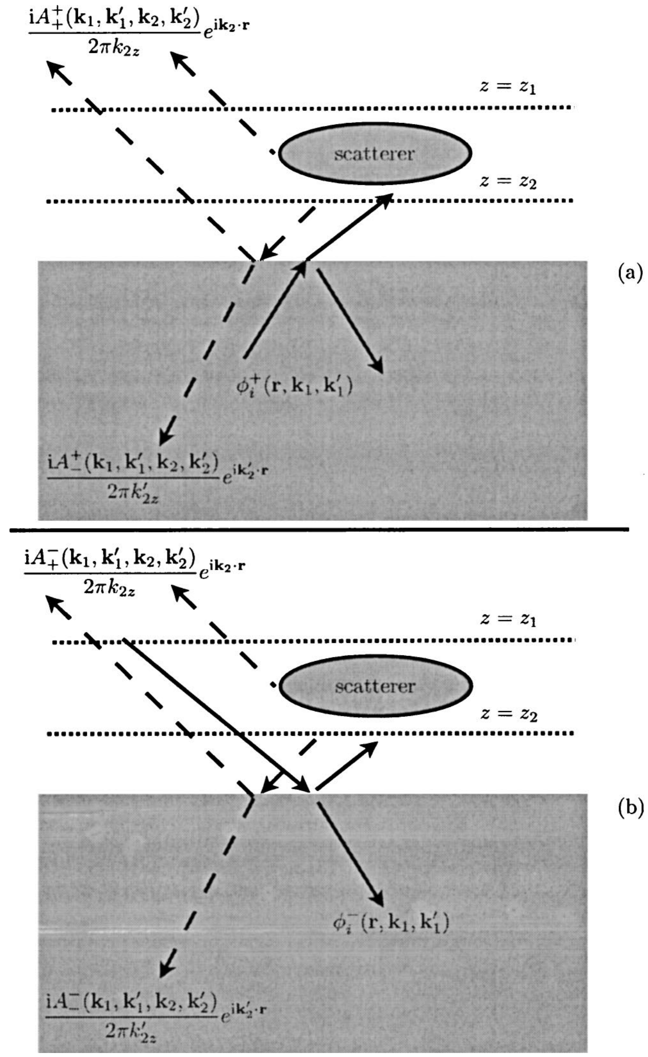

FIG. 2. Illustrating the notation for the half-space problem. In (a), an incident mode $\phi_i^+(\mathbf{r},\mathbf{k}_1,\mathbf{k}_1')$ associated with sources in the lower half space is represented by a solid line indicating the wave vector of the three planewave components of the mode. A mode of the scattered field is represented by a dashed line. The two plane wave components of the scattered field in the upper half space combine to produce an outgoing plane wave with wave vector $\mathbf{k}_2$ and amplitude proportional to $A_+^+(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2')$. The plane wave component of the scattered mode in the lower half space is proportional to $A_-^+(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2')$. In (b), the notation is similarly illustrated, here with a different mode of the incident field, $\phi_i^-(\mathbf{r},\mathbf{k}_1,\mathbf{k}_1')$, generated by sources in the upper half space.

The source of the field is assumed to be entirely outside the region of the scatterer, that is, outside the region $z_1 \geq z \geq z_2$. In the region of the scatterer the incident field obeys Eq. (8) and may be represented as a superposition of modes of this equation analogous to the angular spectrum representation for the free-space problem. We will exclude from our consideration the case that sources are located in the region $z_1 > z > z_2$. Writing $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$, the reflection of the point $\mathbf{r}$ through the $z=0$ plane is given by $\tilde{\mathbf{r}} = \rho\hat{\boldsymbol{\rho}} - z\hat{\mathbf{z}}$. The modes incident from the lower half space may be expressed as

$$\phi_i^+(\mathbf{r};\mathbf{k},\mathbf{k}') = \{\Theta(-z)[e^{i\mathbf{k}'\cdot\mathbf{r}} + R'(\mathbf{k},\mathbf{k}')e^{i\mathbf{k}'\cdot\tilde{\mathbf{r}}}] + \Theta(z)T'(\mathbf{k},\mathbf{k}')e^{i\mathbf{k}\cdot\mathbf{r}}\} \quad (38)$$

and the modes incident from the upper half space are

$$\phi_i^-(\mathbf{r};\mathbf{k},\mathbf{k}') = \{\Theta(z)[e^{i\mathbf{k}\cdot\tilde{\mathbf{r}}} + R(\mathbf{k},\mathbf{k}')e^{i\mathbf{k}\cdot\mathbf{r}}] + \Theta(-z)T(\mathbf{k},\mathbf{k}')e^{i\mathbf{k}'\cdot\tilde{\mathbf{r}}}\}. \quad (39)$$

It may be verified that these modes are orthogonal and that they satisfy the reduced wave equation Eq. (2) and the boundary conditions (32) and (33). Moreover, the Green's function for the case that the points $\mathbf{r}'$ and $\mathbf{r}$ lie, respectively, inside and outside the domain $z_1 > z > z_2$, may be represented as a superposition of these incident modes. This set of modes is thus complete on the space of allowed incident fields. The incident field may then be expressed as

$$\psi_i(\mathbf{r}) = \int d^2k_\parallel [a^+(\mathbf{k})\phi_i^+(\mathbf{r};\mathbf{k},\mathbf{k}') + a^-(\mathbf{k})\phi_i^-(\mathbf{r};\mathbf{k},\mathbf{k}')], \quad (40)$$

or, more compactly,

$$\psi_i(\mathbf{r}) = \sum_{n=\pm} \int d^2k_\parallel\, a^n(\mathbf{k})\phi_i^n(\mathbf{r};\mathbf{k},\mathbf{k}'). \quad (41)$$

For convenience, $\phi^\pm(\mathbf{r};\mathbf{k},\mathbf{k}')$ will denote the total field generated at a point $\mathbf{r}$ by scattering of an incident mode $\phi_i^\pm(\mathbf{r};\mathbf{k},\mathbf{k}')$. As before, the total field modes are given in terms of their incident and scattered parts as $\phi^\pm = \phi_i^\pm + \phi_s^\pm$. The total field may be represented in terms of the total field modes by the expression

$$\psi(\mathbf{r}) = \sum_{n=\pm} \int d^2k_\parallel\, a^n(\mathbf{k})\phi^n(\mathbf{r};\mathbf{k},\mathbf{k}') \quad (42)$$

and the scattered field by

$$\psi_s = \sum_{n=\pm} \int d^2k_\parallel\, a^n(\mathbf{k})\phi_s^n(\mathbf{r};\mathbf{k},\mathbf{k}'). \quad (43)$$

The extinguished power [Eq. (15)] may thus be expressed as

$$P_e = 4\pi k\, \mathrm{Im} \int d^3r \int d^2k_{1\parallel} d^2k_{2\parallel} \times \sum_{m,n=\pm} a^{m*}(\mathbf{k}_2)a^n(\mathbf{k}_1)\phi_i^{m*}(\mathbf{r};\mathbf{k}_2,\mathbf{k}_2')\phi^n(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1')\eta(\mathbf{r}). \quad (44)$$

The scattered field modes $\phi_s^\pm$ may be expressed in the form of angular spectra as in Eq. (23). The situation is somewhat more complicated now, and some explanation of the notation is appropiate. The amplitude A is now a function of the wave vectors corresponding to the transverse wave vector of the incident mode and the plane wave into which that mode is scattered. That is, $A_+^\pm(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2')$ is the amplitude for the scattering of the incident mode $\phi_i^\pm(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1')$ into the outgoing (in the upper half space) plane wave $\exp(i\mathbf{k}_2\cdot\mathbf{r})$, and $A_-^\pm(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2')$ is the amplitude for the scattering of the incident mode $\phi_i^\pm(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1')$ into the outgoing (in the lower half space) plane wave $\exp(i\mathbf{k}_2'\cdot\tilde{\mathbf{r}})$. Then in the upper half space the scattered mode may be expressed as

$$\phi_s^\pm(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1') = \frac{i}{2\pi}\int \frac{d^2k_{2\parallel}}{k_{2z}} A_+^\pm(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2')e^{i\mathbf{k}_2\cdot\mathbf{r}} \quad (45)$$

and in the lower half-space,

$$\phi_s^\pm(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1') = \frac{i}{2\pi}\int \frac{d^2k_{2\parallel}}{k_{2z}'} A_-^\pm(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2')e^{i\mathbf{k}_2'\cdot\tilde{\mathbf{r}}} \quad (46)$$

The normalization has been chosen so that in the upper half space

$$\phi_s^\pm(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1') \sim \frac{e^{ik_0r}}{r}A_+^\pm(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2'),\ k_0r\to\infty, \quad (47)$$

where $\mathbf{k}_2$ is parallel to $\mathbf{r}$. In the lower half space,

$$\phi_s^\pm(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1') \sim \frac{e^{ink_0r}}{r}A_-^\pm(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\ \mathbf{k}_2'),\ k_0r\to\infty, \quad (48)$$

where $\mathbf{k}_2' \parallel \tilde{\mathbf{r}}$.

The scattering amplitudes may be determined by considering, for the upper half space, the scattered field in some plane $z=z_1$ where z_1 is chosen so that the susceptibility of the scatterer is zero for $z > z_1$. Then

$$A_+^\pm(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2') = \frac{-ik_{2z}e^{-ik_{2z}z_1}}{2\pi}\int_{z=z_1} d^2r\, e^{-i\mathbf{k}_{2\parallel}\cdot\mathbf{r}}\phi_s^\pm(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1'). \quad (49)$$

The scattering amplitude in the lower half space may be determined by considering the scattered field in some plane $z=z_2\leq 0$,

$$A_-^\pm(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2') = \frac{-ik_{2z}'e^{ik_{2z}'z_2}}{2\pi}\int_{z=z_2} d^2r\, e^{-i\mathbf{k}_{2\parallel}\cdot\mathbf{r}}\phi_s^\pm(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1'). \quad (50)$$

The scattered field satisfies the integral equation

$$\phi_s^\pm(\mathbf{r};\mathbf{k},\mathbf{k}') = k_0^2\int d^3r'\, G(\mathbf{r},\mathbf{r}')\eta(\mathbf{r}')\phi^\pm(\mathbf{r}';\mathbf{k},\mathbf{k}'). \quad (51)$$

Making use of Eqs. (49)–(51), one finds that

$$A_+^\pm(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2') = k_0^2\int d^3r\left[e^{-i\mathbf{k}_2\cdot\mathbf{r}} + R(\mathbf{k}_2,\mathbf{k}_2')e^{-i\mathbf{k}_2\cdot\tilde{\mathbf{r}}}\right] \times\eta(\mathbf{r})\phi^\pm(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1'), \quad (52)$$

and

$$A_-^\pm(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2,\mathbf{k}_2') = k_0^2\frac{k_{2z}'}{k_{2z}}\int d^3r\, T(\mathbf{k}_2,\mathbf{k}_2')e^{-i\mathbf{k}_2\cdot\tilde{\mathbf{r}}} \times\eta(\mathbf{r})\phi^\pm(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1'). \quad (53)$$

The integrals in Eq. (44) may be identified with the scattering amplitude in certain directions. It is useful to note that in the region of the scatterer, i.e., $z' > 0$,

$$\phi_i^{+*}(\mathbf{r};\mathbf{k},\mathbf{k}') = T'^*(\mathbf{k},\mathbf{k}')[e^{-i\mathbf{k}^*\cdot\mathbf{r}} + R(\mathbf{k}^*,\mathbf{k}'^*)e^{-i\mathbf{k}^*\cdot\tilde{\mathbf{r}}}] + R'^*(\mathbf{k},\mathbf{k}')T'(\mathbf{k}^*,\mathbf{k}'^*)e^{-i\mathbf{k}^*\cdot\tilde{\mathbf{r}}} \tag{54}$$

and

$$\phi_i^{-*}(\mathbf{r};\mathbf{k},\mathbf{k}') = R^*(\mathbf{k},\mathbf{k}')[e^{-i\mathbf{k}^*\cdot\mathbf{r}} + R(\mathbf{k}^*,\mathbf{k}'^*)e^{-i\mathbf{k}^*\cdot\tilde{\mathbf{r}}}] + T'^*(\mathbf{k},\mathbf{k}')T(\mathbf{k}^*,\mathbf{k}'^*)e^{-i\mathbf{k}^*\cdot\tilde{\mathbf{r}}}, \tag{55}$$

where we have made use of the identity that $T^*(\mathbf{k},\mathbf{k}') = T(\mathbf{k}^*,\mathbf{k}'^*)$, and similar expressions for R and R'. By substituting these expressions into Eqs. (52) and (53) and comparing to Eq. (15), the power extinguished from an incident field $\phi_i^+(\mathbf{k},\mathbf{k}',\mathbf{r})$ is seen to be given by the expression

$$P_e = \frac{4\pi}{k_0}\,\mathrm{Im}[T'^*(\mathbf{k},\mathbf{k}')A_+^+(\mathbf{k},\mathbf{k}',\mathbf{k}^*,\mathbf{k}'^*) + R'^*(\mathbf{k},\mathbf{k}')A_-^+(\mathbf{k},\mathbf{k}',\mathbf{k}^*,\mathbf{k}'^*)]. \tag{56}$$

This result has a clear physical interpretation: In the absence of the scatterer, the incident field imparts a certain amount of power to the far zone via the outgoing plane waves reflected from and transmitted through the boundary of the half spaces. The scatterer depletes, or extinguishes, some of the power from the incident field. In order to properly account for the total power, the field produced on scattering must interfere coherently with the incident field in order to extinguish that field. Thus the incident mode $\phi_i^+(\mathbf{r};\mathbf{k}_1,\mathbf{k}_1')$ delivers power to the far zone through the plane waves $e^{i\mathbf{k}_1'\cdot\tilde{\mathbf{r}}}$ and $e^{i\mathbf{k}_1\cdot\mathbf{r}}$, and the extinguished power is directly related to the amplitude of the scattered plane waves in those same directions as may be seen in expression (56). The situation is illustrated in Fig. 2. In the event that the incident mode consists of a wave totally internally reflected in the $z<0$ half space, the extinguished power is related to the amplitude of the modes of the field coupled back into the $z<0$ half space, propagating in the direction of the beam reflected from the interface.

The power extinguished from the incident field $\phi_i^-(\mathbf{r};\mathbf{k},\mathbf{k}')$ is given by the expression

$$P_e = \frac{4\pi}{k_0}\,\mathrm{Im}[T^*(\mathbf{k},\mathbf{k}')A_-^-(\mathbf{k},\mathbf{k}',\mathbf{k}^*,\mathbf{k}'^*) + R^*(\mathbf{k},\mathbf{k}')A_+^-(\mathbf{k},\mathbf{k}',\mathbf{k}^*,\mathbf{k}'^*)]. \tag{57}$$

This expression may also be interpreted as relating the extinguished power to the amplitude of the scattered waves which are coincident with the outgoing parts of the incident field. In general, with an incident field given by Eq. (41),

$$\begin{aligned} P_e = \sum_{n=\pm}\frac{4\pi}{k_0}\,\mathrm{Im}\Bigg\{ &\int d^2k_{1\|}d^2k_{2\|}\, a^n(\mathbf{k}_1)a^{+*}(\mathbf{k}_2) \\ &\times[T'^*(\mathbf{k}_2,\mathbf{k}_2')A_+^n(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2^*,\mathbf{k}_2'^*) \\ &+ R'^*(\mathbf{k}_2,\mathbf{k}_2')A_-^n(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2^*,\mathbf{k}_2'^*)] \\ &+ \int d^2k_{1\|}d^2k_{2\|}\, a^n(\mathbf{k}_1)a^{-*}(\mathbf{k}_2) \\ &\times[T^*(\mathbf{k}_2,\mathbf{k}_2')A_-^n(\mathbf{k}_1,\mathbf{k}_1',\mathbf{k}_2^*,\mathbf{k}_2'^*) \\ &+ R^*(\mathbf{k}_2,\mathbf{k}_2')A_+^n(\mathbf{k}_1,\mathbf{k}_1,\mathbf{k}_2'^*,\mathbf{k}_2'^*)]\Bigg\}. \end{aligned} \tag{58}$$

V. DISCUSSION

The results presented here provide insight into the interference mechanisms that ensure energy conservation in the scattering of scalar waves. Equation (15) provides a framework in which to obtain a relationship between the scattering amplitude and the extinguished power in problems with an arbitrary background medium. We have obtained such a relationship for the case that the background medium consists of a lossless half space of index different from the vacuum. This problem is relevant to the scattering of a single evanescent plane wave. Such a field can be generated only in the half-space geometry. In free space, evanescent modes may be present in superposition with other modes of the field if the source is placed near the scatterer, but a single evanescent mode is never present in isolation [7]. The results have a clear physical meaning, namely, that the extinguished power is simply related to the scattering amplitude of the scattered field in the direction of the outgoing plane wave components of the incident field.

To lowest order in the susceptibility, when the scatterer is in vacuum, the extinguished power is the projection of the incident intensity on the imaginary part of the susceptibility,

$$P_e = 4\pi k_0\int_V d^3r|\psi_i(\mathbf{r})|^2\,\mathrm{Im}\,\eta(\mathbf{r}) + O(\eta^2). \tag{59}$$

This formula suggests a manner in which object structure may be investigated. If the intensity of the incident field forms the kernel of a transformation that can be inverted, then the object structure, as described by Im $\eta(\mathbf{r})$, may be found from power extinction measurements. In Refs. [8,9] the incident field was taken to consist of two plane waves and consequently the extinguished power was related to a Fourier transform [8] or to a Fourier-Laplace transform [9] of the object. The present result shows that the extinguished power is a meaningful measure of object structure for certain forms of the incident field. Two tomographic modalities currently in practice may also be understood as variants of this approach. Computed tomography with x rays [14] is accomplished with measurements of the field attenuation along rays passing through the sample under investigation. That situation is described by Eq. (59) when the incident field is assumed to be localized to the ray path. The technique of transmission mode confocal imaging [15] may also be understood in the context of Eq. (59). There the field intensity is much higher at the focus than at any other point in the medium, and, as is well known, the data represent a convolution of the object structure with the intensity in the focus. Because the transmitted field is collected in a confocal arrangement, the signal is simply related to the extinguished power.

Results analogous to those presented here may be obtain for the vector (electromagnetic) field and will be presented in another paper. Layer structures and waveguide geometries will also be studied in future work.

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