Optimal apodizations for finite apertures

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A method is presented for determining the aperture apodization functions needed to optimize any given product of powers of the even-order moments of the beam intensity in the near and far zones. The results are a generalization of previous work [Pure Appl. Opt. **7**, 1221 (1998)] that dealt only with the far-zone moments. These methods are applied to the problem of optimizing the so-called beam propagation factor, $M_P{}^2$. [S0740-3232(99)01107-2]

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The characterization of laser beams by a single parameter, such as the so-called beam propagation factor (sometimes called the quality factor)[1,2] or by a set of numbers such as the moments of the field intensity in the aperture plane and the far zone,[3,4] is useful in optics and engineering.[5] Which method of characterization is most appropriate in particular situations has been frequently discussed in the literature (see, for example, Ref. 6). In this paper we present a method to optimize any product of powers of moments of the field intensity in the aperture plane and in the far zone of the beam. Mixed moments of this kind appear in equations for the propagation of the kurtosis parameter, which has been suggested as another method of beam characterization.[7] There is also a closely related class of optimization problems, the so-called Luneburg problems[8] relating to issues of resolution limits in optical imaging, where these techniques may prove useful. The method introduced in this paper is a generalization of methods used recently to optimize products of even moments of the field in the far zone.[9] We apply these new results to the particular problem of minimizing the beam propagation factor that was introduced by Siegman.[1]

For simplicity, we will restrict our discussion to a beam with one transverse degree of freedom, which we denote x in the aperture plane; similar results hold for two transverse degrees of freedom. The field in the aperture plane will be specified by a function $f(x)$ that vanishes identically outside the range $-\delta < x < \delta$. The transverse coordinate in the far zone will be denoted s, and the field in the far zone will be denoted $F(s)$. Within the accuracy of the paraxial approximation, $F(s)$ is related to the field $f(x)$ in the aperture, apart from a constant multiplicative factor, by the Fourier transform[10]

$$F(s) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi i s x) \mathrm{d}x. \tag{1}$$

The moments of the field intensity of order k in the aperture plane are defined by the expression

$$\langle x^k \rangle \equiv \mathfrak{N} \int x^k |f(x)|^2 \mathrm{d}x, \tag{2}$$

and in the far zone

$$\langle s^k \rangle \equiv \mathfrak{N} \int s^k |F(s)|^2 \mathrm{d}s, \tag{3}$$

where

$$\mathfrak{N}^{-1} \equiv \int |F(s)|^2 \mathrm{d}s = \int |f(x)|^2 \mathrm{d}x. \tag{4}$$

The even-order moments in the far zone may be expressed in the form

$$\langle s^{2k} \rangle = \left(\frac{1}{2\pi} \right)^{2k} \mathfrak{N} \int |f^{(k)}(x)|^2 \mathrm{d}x, \tag{5}$$

where $f^{(k)}$ denotes the kth derivative.

It is our aim to optimize a functional of the form

$$\mathcal{J} = \mathfrak{N}^{\xi} \prod_{k=1}^{N_s} \langle s^{2m_k} \rangle^{\mu_k} \prod_{l=1}^{N_x} \langle x^{2n_l} \rangle^{\nu_l}. \tag{6}$$

Using a calculus-of-variations approach to find extremal solutions for the aperture field, one finds that the first variation of this functional is given by the expression (see Appendix A)

$$\delta \mathcal{J} = \mathfrak{N} \mathcal{J} \left[\sum_{k}^{N_s} \frac{\mu_k \int f^{(m_k)}(x) \delta f^{(m_k)}(x) \mathrm{d}x}{(2\pi)^{2m_k} \langle s^{2m_k} \rangle} + \int \left(\sum_{l}^{N_x} \frac{\nu_l x^{2n_l}}{\langle x^{2n_l} \rangle} - \gamma \right) f(x) \delta f(x) \mathrm{d}x \right], \tag{7}$$

where

$$\gamma \equiv \sum_{k}^{N_s} \mu_k + \sum_{l}^{N_x} \nu_l + \xi. \tag{8}$$

It has been shown that in order for a moment of order $2k$ in the far zone, $\langle s^{2k} \rangle$, to exist, the field in the aperture must be k times differentiable in the aperture, and the field and its derivatives up to and including the $k-1$st derivative must continuously approach zero at the edges of the aperture.[9] Assuming that these conditions are met, the integrals under the first summation in Eq. (7) may be evaluated by integration by parts, and then one obtains the Euler–Lagrange differential equation for the field in the aperture[11]:

$$\sum_{k}^{N_s} \frac{(-1)^{m_k}\mu_k}{(2\pi)^{2m_k}\langle s^{2m_k}\rangle} f^{(2m_k)}(x) + \left(\sum_{l}^{N_x} \frac{\nu_l x^{2n_l}}{\langle x^{2n_l}\rangle} - \gamma\right) f(x) = 0. \tag{9}$$

With use of this differential equation and the boundary conditions required for the existence of the moments in the far zone, it is in principle possible to construct the aperture function that gives a stationary value of the quantity in Eq. (6).

As an example, we apply this method to the beam propagation factor $M_P{}^2$ defined by the expression

$$M_P{}^2 = 4\pi\sqrt{\langle s^2\rangle\langle x^2\rangle}, \tag{10}$$

where the coordinates have been chosen so that $\langle x\rangle = \langle s\rangle = 0$. The resulting differential equation may be written in the form

$$f^{(2)}(x) + \left(\frac{1}{2}M_P{}^2 L^{-2} - \frac{1}{4}L^{-4}x^2\right) f(x) = 0, \tag{11}$$

where

$$L^2 \equiv \frac{1}{4\pi}\sqrt{\langle x^2\rangle/\langle s^2\rangle}. \tag{12}$$

By making the substitution $x = Ly$ and $g(y) = f(Ly)$, we obtain the parabolic cylinder equation, also known as the Weber–Hermite differential equation (Ref. 12, p. 281):

$$g^{(2)}(y) + \left(\frac{1}{2}M_P{}^2 - \frac{1}{4}y^2\right) g(y) = 0. \tag{13}$$

The solutions of Eq. (13) may be expressed in terms of the Hermite functions of order $r = (M_P{}^2 - 1)/2$. If r is *not* a nonnegative integer, the functions $H_r(z)$ and $H_r(-z)$ are linearly independent. We will make this assumption (that r is not a nonnegative integer) because the minimum value of $M_P{}^2 = 1$ is obtained by taking the field to be a Gaussian in an infinite aperture, and any field emerging from a finite aperture will necessarily have a larger value of $M_P{}^2$. Furthermore, we know of solutions in the finite aperture that have $M_P{}^2 < 3$ (see Ref. 9), and so $0 < r < 1$.

The general solutions of Eq. (11) may be expressed in the form

$$f(x) = \alpha S(x) + \beta A(x), \tag{14}$$

where

$$S(x) = \exp(-x^2/4L^2)\left[H_r\left(\frac{x}{L\sqrt{2}}\right) + H_r\left(\frac{-x}{L\sqrt{2}}\right)\right], \tag{15}$$

$$A(x) = \exp(-x^2/4L^2)\left[H_r\left(\frac{x}{L\sqrt{2}}\right) - H_r\left(\frac{-x}{L\sqrt{2}}\right)\right]. \tag{16}$$

The boundary conditions $f(\pm\delta) = 0$ lead to a set of equations that have nontrivial solutions only when the transcendental equation

$$H_r\left(\frac{\delta}{L\sqrt{2}}\right) = \pm H_r\left(\frac{-\delta}{L\sqrt{2}}\right) \tag{17}$$

is satisfied. When this equation is satisfied, either $\alpha = 0$ (upper sign) or $\beta = 0$ (lower sign).

To summarize the example to this point in the calculation: We have found the general form of the functions that make stationary the propagation factor defined in Eq. (10) for any given value of L defined in Eq. (12). There are still two unknown parameters in Eq. (17), namely, L and M_P. The equations must be solved computationally; the minimum values of $M_P{}^2$ as a function of δ/L are shown in Fig. 1. For a given aperture size, the $M_P{}^2$ factor may be made arbitrarily close to unity by choosing L to be sufficiently small. This amounts to making the effective width of the aperture at half-maximum transmission (FWHM) very small compared with the total width of the aperture, thus masking the edges. In this limit, it is practically impossible to distinguish the aperture of finite extent from the aperture of infinite extent, because the edges of the aperture are then far into the tails of the Gaussian distribution appearing in Eqs. (15) and (16). Also in this limit the beam is no longer paraxial, as we will now show.

We note that $(\lambda/2\pi)^2\langle s^2\rangle$ represents the angular spread of the beam in the far zone, where λ is the wavelength of the field. In order that the field be beamlike, the angular spread of the beam must be sufficiently small that the paraxial approximation remains valid, i.e., $\langle s^2\rangle \ll (2\pi/\lambda)^2$. Since M_P is bounded from below by unity, we must have that

$$L \gg \frac{\lambda}{2\pi}. \tag{18}$$

Thus to ensure that the field is beamlike, we require that $L = L_{\min}$ where $L_{\min}$ is some nonzero constant that satisfies inequality (18). This restriction does not change the form of the differential equation derived by the variational method. If L does not satisfy inequality (18), then the beam spreads appreciably in the far zone and is nonparaxial.

Figure 1 indicates that the propagation factor may still be made arbitrarily close to unity for appropriately large values of δ. This is what one would intuitively expect, because the infinite-aperture result should be recovered as $\delta \to \infty$. For example, by taking $\delta/L_{\min} = 3$ we find that the minimum propagation factor is $M_P{}^2 = 1.048$,

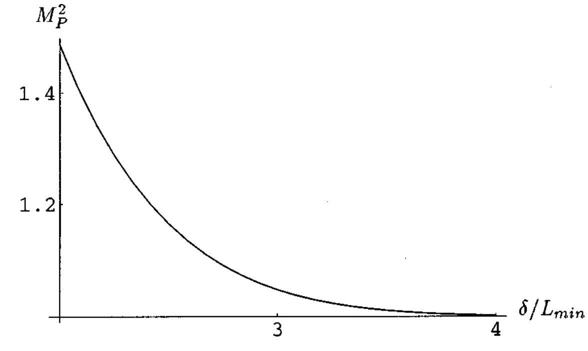

Fig. 1. Propagation factor $M_P{}^2$ as a function of $\delta/L_{\min}$.

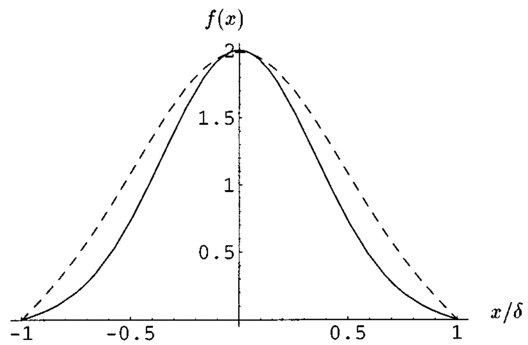

Fig. 2. Aperture functions that minimize $M_P{}^2$ when $\delta/L_{\min} = 3$ (dashed curve) and when $\delta/L_{\min} = 4$ (solid curve).

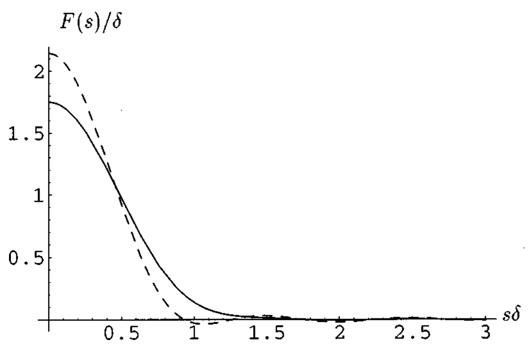

Fig. 3. Far-zone distributions resulting from the aperture functions that minimize $M_P{}^2$ when $\delta/L_{\min} = 3$ (dashed curve) and when $\delta/L_{\min} = 4$ (solid curve).

and with $\delta/L_{\min} = 4$ it has the value $M_P{}^2 = 1.002$. The aperture functions (with $\alpha = 1$, $\beta = 0$) corresponding to each case are shown in Fig. 2, and the corresponding far-zone distributions are shown in Fig. 3.

We conclude by saying that we have extended the earlier method of determining aperture functions that minimize the products of the moments of the field of a beam in the far zone to include products of the moments of the field in the aperture plane. We have applied this method to the problem of minimizing the so-called beam propagation factor, and in the process we have introduced a new parameter, L [Eq. (12)], which represents the angular spread of the beam and should be specified along with the propagation factor M_P to give a more accurate description of the beam.

APPENDIX A

In this appendix we provide a brief derivation of Eq. (7). For a rigorous justification of some of the steps taken in this derivation, the reader is referred to Ref. 13.

Beginning with the functional $\mathcal{J}$ defined in Eq. (6), we may write its explicit dependence on the function $f(x)$, using Eqs. (2), (5), and (6):

$$\mathcal{J}[f] = \frac{1}{\left[\int |f(x)|^2 \mathrm{d}x\right]^{\gamma}} \prod_{k=1}^{N_s} \left[\frac{1}{(2\pi)^2}\int |f^{(m_k)}(x)|^2 \mathrm{d}x\right]^{\mu_k} \times \prod_{l=1}^{N_x} \left[\int x^{2n_l}|f(x)|^2 \mathrm{d}x\right]^{\nu_l}, \tag{A1}$$

where γ is given in Eq. (8)

We seek a function $f(x)$ that makes $\mathcal{J}$ stationary. Consider functions $q(x)$ infinitesmally different from $f(x)$, such that

$$q(x) = f(x) + \epsilon\delta f(x), \tag{A2}$$

where ϵ is a dimensionless parameter and $\delta f(x)$ is a function that satisfies the convergence criteria.[9] If $\mathcal{J}$ is stationary with respect to variations of f, then the first variation of $\mathcal{J}$, $\delta\mathcal{J}$, must be identically zero; i.e., it must satisfy the relation

$$\delta\mathcal{J} \equiv \left.\frac{\partial \mathcal{J}[q]}{\partial \epsilon}\right|_{\epsilon=0} = 0. \tag{A3}$$

That is, if $\mathcal{J}$ is stationary at f in the space of functions, then $\epsilon = 0$ is a stationary point of $\mathcal{J}$ with respect to ϵ.

If we substitute from Eq. (A2) into Eq. (A1) and take the derivative as indicated in Eq. (A3), Eq. (7) follows.

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