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The Relationship of Transform Coefficients for Differing Transforms and/or Differing Subblock Sizes

Brynmor J. Davis and S. Hamid Nawab

***Abstract*—Jiang and Feng have developed a relationship between the discrete cosine transform (DCT) coefficients of a block and those of its subblocks. Their derivation of this result can be significantly simplified. The new derivation also generalizes to all linear, invertible transforms and any separable subblock geometry.**

***Index Terms*—Separable transform, subblock transform, transform coefficient relationship, transform domain processing.**

I. INTRODUCTION

A relationship between the discrete cosine transform (DCT) [1] coefficients of an image and the DCT coefficients of its subblocks has been derived by Jiang and Feng [2]. This direct relationship is a computationally efficient alternative to first applying an inverse DCT (to obtain the image in the pixel domain) and then taking the forward DCT over the subblocks. Direct coefficient manipulations such as this facilitate image processing in the transform domain, which is currently an active area of research [3]–[6].

The derivation contained in [2] can be significantly simplified using a matrix representation of the transforms. In addition, this approach allows the relationship to be generalized from the DCT to any linear, invertible transform (and even certain mixes of transforms across the subblocks). The constraints on the geometries of the subblocks are also relaxed—while [2] only deals with $A:1$ ratios between the subblocks (e.g., a 12 × 20 image being divided into four 6 × 10 subblocks), the new derivation can be applied to $A:B$ ratios (e.g. four 6 × 10 subblocks being related to 15 4 × 4 subblocks) or even certain nonuniform ratios (e.g. four 6 × 10 subblocks being related to a 12 × 8 subblock and a 12 × 12 subblock).

A related approach is nonuniform transform domain filtering (NTDF) [7], which is an extension of transform domain filtering (TDF) [8]. Theoretically, NTDF can be used to relate coefficients where there is a constant $A : B$ ratio between the subblock sizes. However, [7] does not include a clear relation between the coefficients, nor does it describe explicitly which subblock and transform geometries can be handled. Additionally, NTDF is designed specifically for a pipelining architecture.

In comparison, the results presented here give a fundamental theoretical relation, along with a well-defined set of sufficient conditions for its validity. These conditions are more general than those described in either [2] or [7]. The result is presented first in one dimension and then in two dimensions through the use of separable transforms and separable subblock geometries. An example is also provided.

II. ONE-DIMENSIONAL RESULT

The majority of useful discrete transforms are linear and invertible (e.g., the DCT, the discrete Fourier transform (DFT) [9], and the discrete wavelet transform (DWT) [10]). This means their operation on a $N \times 1$ vector x can be represented by a matrix multiplication (see [9] for examples). The transform matrix will be $N \times N$ and invertible. The result of this matrix multiplication gives a vector of transform coefficients.

$$X = Tx \tag{1}$$
$$x = T^{-1}X. \tag{2}$$

If the vector x is split into P contiguous subvectors with lengths $N_1, N_2, \cdots, N_P$ ($\sum_{i=1}^{P} N_i = N$) and a (possibly different) transform applied to each subvector, then the resulting set of transform coefficients can still be found through multiplication with a $N \times N$ matrix $\hat{T}$.

$$\hat{X} = \hat{T}x \tag{3}$$
$$x = \hat{T}^{-1}\hat{X}. \tag{4}$$

The $\hat{}$ indicates that transforms have been taken over a set of subvectors. In addition, the matrices $\hat{T}$ and $\hat{T}^{-1}$ can be constructed from the transform matrices for each subvector. They will exhibit a block diagonal structure, which is shown as

$$\hat{T} = \begin{bmatrix} T_1 & 0 & & \cdots & 0 \\ 0 & T_2 & 0 & \cdots & 0 \\ 0 & 0 & T_3 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & T_P \end{bmatrix} \tag{5}$$

$$\hat{T}^{-1} = \begin{bmatrix} T_1^{-1} & 0 & & \cdots & 0 \\ 0 & T_2^{-1} & 0 & \cdots & 0 \\ 0 & 0 & T_3^{-1} & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & T_P^{-1} \end{bmatrix}. \tag{6}$$

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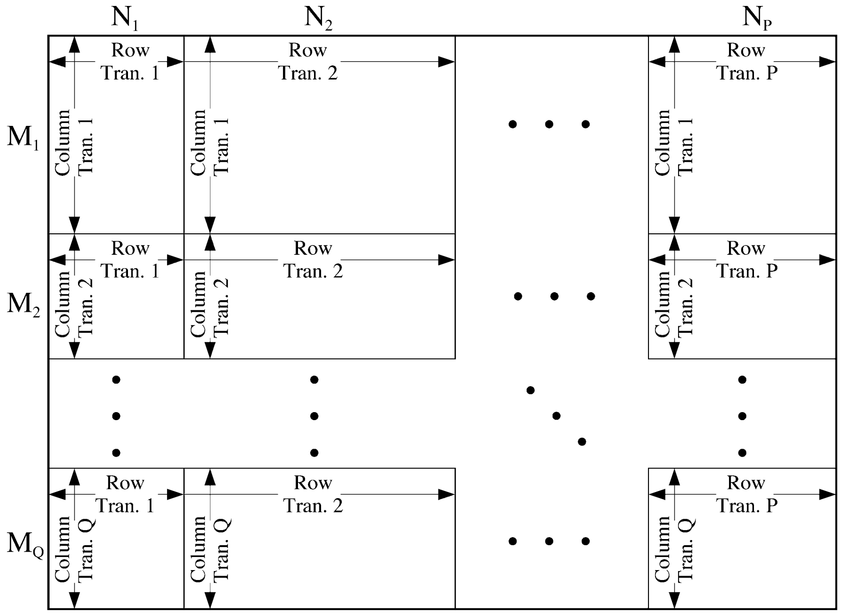

Fig. 1. Form of separable subblock transforms.

Note that the matrix T_i will be $N_i \times N_i$ in order to represent the N_i-point transform taken over the i^{th} subvector.

Now, suppose that two different methodologies are used in taking these subvector transforms (i.e. different transforms and/or different partitions of x).

$$\hat{X}_1 = \hat{T}_1 x \tag{7}$$
$$\hat{X}_2 = \hat{T}_2 x. \tag{8}$$

The objective is to find a direct relationship between $\hat{X}_1$ and $\hat{X}_2$. A direct relationship is a single step between these two coefficient vectors. The alternative is first applying an inverse transform to one coefficient vector (resulting in x) and then applying a forward transform to find the other coefficient vector.

For linear invertible transforms, a direct linear relationship always exists and can easily be shown to be unique. Let matrix multiplication by $\hat{R}_{12}$ take $\hat{X}_1$ to $\hat{X}_2$ and $\hat{R}_{21}$ take $\hat{X}_2$ to $\hat{X}_1$. These relationships are found as follows:

$$\begin{aligned}\hat{X}_2 &= \hat{T}_2 x\\ &= \hat{T}_2\left(\hat{T}_1^{-1}\hat{X}_1\right)\\ &= \hat{R}_{12}\hat{X}_1\end{aligned} \tag{9}$$

where

$$\hat{R}_{12} = \hat{T}_2\hat{T}_1^{-1}. \tag{10}$$

Similarly

$$\hat{X}_1 = \hat{R}_{21}\hat{X}_2 \tag{11}$$

where

$$\hat{R}_{21} = \hat{R}_{12}^{-1} = \hat{T}_1\hat{T}_2^{-1}. \tag{12}$$

Thus, the direct relationship between the two sets of transform coefficients is found by precomputing the matrix product of the matrix representing the forward transformation (from the output coefficient set) with the matrix representing the inverse transform (from the input coefficient set). The subvectors can be any partition of x, and the only restrictions on the subvector transforms are that each one must be linear and invertible (this includes the useful cases of the DFT, the DCT, the DWT, and the identity transform).

III. Two-Dimensional Result

In general, operators on two-dimensional (2-D) signals can also be represented as matrices. This process involves ordering the elements of the image and converting it to a one-dimensional (1-D) vector (see [9] for a description of this process). Theoretically, a direct linear relation could be derived using this formulation. However, the matrices involved would become prohibitively large. This difficulty can be avoided by using separable transforms (e.g. the 2D-DFT and the 2D-DCT).

Separable 2-D transforms can be decomposed into a 1-D transform on each column followed by a 1-D transform on each row of the result. Thus, if the matrix T performs the transformation to be used on the columns, the matrix U performs the transformation to be used on the rows, and the 2-D signal x is a $N \times M$ matrix, the 2-D transform coefficients can be found as follows (linearity and invertibility are again assumed):

$$X = TxU^T \tag{13}$$
$$x = T^{-1}X(U^{-1})^T \tag{14}$$

where X will be a $N \times M$ matrix, T is a $N \times N$ matrix, and U is a $M \times M$ matrix.

If a pair of transform matrices ($\hat{T}$ and $\hat{U}$) is defined as in (5), then the resulting coefficients will be the transforms of separable subblocks of the image. A separable subblock transform has a regular structure to its boundaries, and its row-column transforms as shown in Fig. 1.

If subblock transforms of an image are taken in two different ways (defined by $\hat{T}_1, \hat{U}_1, \hat{T}_2$ and $\hat{U}_2$), a linear relation between the two can be found. The two sets of transform coefficients are given by the following equations:

$$\hat{X}_1 = \hat{T}_1 x \hat{U}_1^T \tag{15}$$
$$\hat{X}_2 = \hat{T}_2 x \hat{U}_2^T. \tag{16}$$

From (10) and (12), the following matrices can be defined:

$$\hat{R}_{12} = \hat{T}_2\hat{T}_1^{-1} \tag{17}$$
$$\hat{R}_{21} = \hat{T}_1\hat{T}_2^{-1} \tag{18}$$
$$\hat{S}_{12} = \hat{U}_2\hat{U}_1^{-1} \tag{19}$$
$$\hat{S}_{21} = \hat{U}_1\hat{U}_2^{-1}. \tag{20}$$

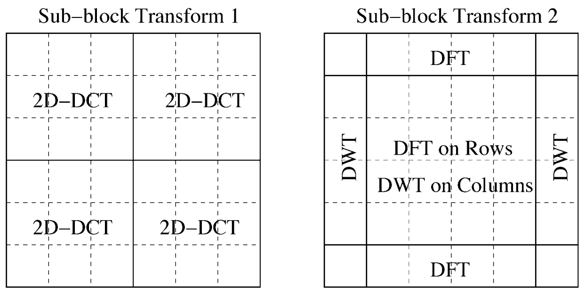

Fig. 2. Example subblock transforms.

These matrices give the relationship between the two sets of 2-D transform coefficients $\hat{X}_1$ and $\hat{X}_2$.

$$\hat{X}_2 = \hat{R}_{12}\hat{X}_1\hat{S}_{12}^T \tag{21}$$

$$\hat{X}_1 = \hat{R}_{21}\hat{X}_2\hat{S}_{21}^T. \tag{22}$$

The equation above demonstrates how separability can be used to cast the 2-D problem as two 1-D problems.

IV. Illustrative Example

This section gives a simple example on a 6×6 image. The two separable subblock transforms chosen are shown in Fig. 2. The first is a simple 2D-DCT transform on each of four 3×3 subblocks. The second subblock transform is more complicated (and somewhat contrived) in order to show the generality of the result. In this case, the transform coefficients are found by taking the DFT and DWT. The corner pixels are treated as four subblocks (with no transform). The DWT is applied to the remaining vertical edge pixels, and the DFT is applied to the remaining horizontal edge pixels. This leaves a 4×4 subblock in the middle. The DFT is applied to the rows of this subblock and the DWT to the columns. In this example, the DWT will be taken using the Haar wavelet [10].

The first step is to construct the matrices of the transforms used [i.e., the T matrix shown in (1)]. These are easily found using the definitions of the DCT, DFT, and DWT. For the first subblock transform, a three-point DCT is the only transform used.

$$T_{\mathrm{DCT}} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{(2\sqrt{3})} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{(2\sqrt{3})} \end{bmatrix} \tag{23}$$

For the second subblock transform, a four-point DFT and a four-point DWT are used.

$$T_{\mathrm{DFT}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{j}{2} & -\frac{1}{2} & \frac{j}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{j}{2} & -\frac{1}{2} & -\frac{j}{2} \end{bmatrix} \tag{24}$$

$$T_{\mathrm{DWT}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{25}$$

The next step is to use these transform matrices to define $\hat{T}_1$, $\hat{U}_1$, $\hat{T}_2$, and $\hat{U}_2$, as in (15) and (16). This is done by examining the row and column structure of the subblock transform and applying (5) appropriately. For this example, the following matrices are constructed:

$$\hat{T}_1 = \begin{bmatrix} & & & 0 & 0 & 0 \\ & T_{\mathrm{DCT}} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & T_{\mathrm{DCT}} & \\ 0 & 0 & 0 & & & \end{bmatrix} \tag{26}$$

$$\hat{U}_1 = \begin{bmatrix} & & & 0 & 0 & 0 \\ & T_{\mathrm{DCT}} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & T_{\mathrm{DCT}} & \\ 0 & 0 & 0 & & & \end{bmatrix} \tag{27}$$

$$\hat{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & 0 \\ 0 & & & & & 0 \\ 0 & & T_{\mathrm{DWT}} & & & 0 \\ 0 & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{28}$$

$$\hat{U}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & 0 \\ 0 & & & & & 0 \\ 0 & & T_{\mathrm{DFT}} & & & 0 \\ 0 & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

The calculation of $\hat{R}_{12}$ and $\hat{S}_{12}$ is easily done using (17) and (19). These matrices are displayed in the following (to two decimal places):

$$\hat{R}_{12} = \begin{bmatrix} 0.58 & 0.71 & 0.41 & 0 & 0 & 0 \\ 0.58 & -0.35 & -0.20 & 0.58 & 0.35 & -0.20 \\ -0.58 & 0.35 & 0.20 & 0.58 & 0.35 & -0.20 \\ 0 & -0.5 & 0.87 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & -0.87 \\ 0 & 0 & 0 & 0.58 & -0.71 & 0.41 \end{bmatrix} \tag{30}$$

and (31), shown at the bottom of the page. $\hat{R}_{12}$ and $\hat{S}_{12}$ allow the coefficients of the second subblock transform to be calculated from those of the first by using (21).

The coefficients of the first subblock transform can also be calculated from those of the second by using (22). This requires $\hat{R}_{21}$ and $\hat{S}_{21}$. These can either be calculated using (18) and (20), or by noticing $\hat{R}_{21} = \hat{R}_{12}^{-1}$ and $\hat{S}_{21} = \hat{S}_{12}^{-1}$ and using (30) and (31).

$$\hat{S}_{12} = \begin{bmatrix} 0.58 & 0.71 & 0.41 & 0 & 0 & 0 \\ 0.58 & -0.35 & -0.20 & 0.58 & 0.35 & -0.20 \\ 0.29 - 0.29j & 0.35j & -0.41 - 0.20j & -0.29 + 0.29j & -0.35 & -0.20 - 0.41j \\ 0 & 0.35 & -0.61 & 0 & 0.35 & 0.61 \\ 0.29 + 0.29j & -0.35j & -0.41 + 0.20j & -0.29 - 0.29j & -0.35 & -0.20 + 0.41j \\ 0 & 0 & 0 & 0.58 & -0.71 & 0.41 \end{bmatrix} \tag{31}$$

V. CONCLUSIONS

A general linear relationship between the coefficients of two different subblock transformations was developed. This relationship holds for any mix of linear, invertible transforms and separable subblock transform geometries (as illustrated in Fig. 1). The relationship can be found by simply precomputing the result of an inverse transform matrix multiplied by a differing forward transform matrix.

This result is a generalization of previous work by Jiang and Feng [2]. In that paper, it was also shown that the matrix giving their linear relation is sparse (which results in reduced computational load). This property holds for the DCT case when the subblocks are $A:1$ ratios of the larger blocks. In general (i.e., for other transforms and subblock ratios), this sparseness may not be present. This means that the reduction in computational load may not be as great. Whether it is efficient to relate transform coefficients by the method developed here will depend on the particular scenario as well as the applicability of any fast algorithms for the transforms of interest (these algorithms may make relating the coefficients through an inverse transform operation followed by a forward transform operation more desirable).

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A Low-Complexity Adaptive Echo Canceller for xDSL Applications

Shou-Sheu Lin and Wen-Rong Wu

***Abstract*—A finite impulse response (FIR)-based adaptive filter structure is proposed for echo cancellation in xDSL applications. The proposed algorithm consists of an FIR filter, a cascaded interpolated FIR filter, and a tap-weight overlapping and nulling scheme. This filter requires low computational complexity and inherits the stable characteristics of the conventional FIR filter. Simulations show that the proposed echo canceller can effectively cancel the echo up to 73.4 dB [for a single-pair high-speed digital subscriber line (SHDSL) system]. About 55% complexity reduction can be achieved compared with a conventional FIR filter.**

***Index Terms*—Adaptive filter, DSL, echo cancellation, interpolated FIR filter.**

I. INTRODUCTION

In a digital subscriber loop (DSL) environment, full duplex transmission via a single twisted pair can be achieved using a hybrid circuit. Due to the impedance mismatch problem, the hybrid circuit will introduce echoes. A typical echo response, shown in Fig. 1, consists of a short and rapidly changing head echo and a long and slowly decaying tail echo. Conventionally, an adaptive transversal FIR filter [1] is used to synthesize and cancel the echo. For high-speed applications such as HDSL [2], HDSL2 [3], and single-pair high-speed digital subscriber line (SHDSL) [4], the echo response is usually very long. The conventional FIR echo canceller may require hundreds of tap weights, and the computational complexity becomes very high.

In order to reduce the computational complexity, some researchers tried to use an adaptive infinite impulse response (IIR) filter to cancel the tail echo. However, the adaptive IIR filtering suffers from the local minima and stability problems. Since an IIR filter usually consists of a feedforward and a feedback filter, a compromising approach is to let the feedforward filter be adaptive only. In [5], August *et al.* collected some echo responses for the European subscriber loops and used a criterion to determine the feedback filter optimally. In [6], Gordon *et al.* considered echo cancellation as a series expansion problem. They used a set of IIR orthonormal functions to expand the echo response and let the expanding coefficients be adaptive. The orthonormal responses were obtained using a set of predetermined cascaded feedback filters. If only a small number of loops are considered, good performance can be obtained using these methods. However, since the existing loop responses are versatile, it will be difficult to find a feedback filter that will always yields the optimal performance.

To retain the FIR structure of the echo canceller, and to reduce the complexity, an interesting echo canceller structure was proposed in [7]. The canceller is cascaded from an adaptive FIR head echo canceller and an adaptive interpolated FIR (IFIR) tail echo canceller. Since the tail echo always decays smoothly, an IFIR filter with a small number of coefficients can effectively cancel the echo. Unfortunately, the IFIR filter proposed in [7] has an uncontrollable transient response, and the direct cascade of an FIR and an IFIR filter will leave a certain period of the echo response uncancelled. Although this problem is critical,

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